

Special test configuration and K-stability of Fano varieties

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For any flat projective family $(\mathcal{X}, \mathcal{L}) \rightarrow C$ such that the generic fibre \mathcal{X}_η is a klt \mathbb{Q} -Fano variety and $\mathcal{L}|_{\mathcal{X}_\eta} = -rK_\eta$, we use the techniques from the minimal model program (MMP) to modify the total family. The end product is a family such that *every* fiber is a klt \mathbb{Q} -Fano variety. Moreover, we can prove the Donaldson-Futaki intersection numbers of the appearing models *decreases*. When the family is a test configuration of a fixed Fano variety $(X, -rK_X)$, this implies Tian's conjecture: given X a Fano manifold, to test its K-(semi)stability, we only need to test on the special test configurations.

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1 Introduction and main results

Through out this paper, we work over the field of complex numbers.

1.1 Degenerations of Fano varieties

For various questions, especially for compactifying the moduli spaces, people are interested in the degenerations of varieties. When the varieties have positive canonical classes, i.e., they are *canonically polarized*, this question has attracted many interests. The one dimensional case, namely, the degeneration of smooth curves of genus at least 2, has been understood well after Deligne-Mumford's work [DM69]. The study of higher dimensional case by an analogue strategy was initiated more than two decades ago (see [KSB88], [Ale94]). Using the recent monumental progress on the minimal model program of [BCHM10] and many other work, the fundamental aspects of this theory are close to be completely settled. See Kollár's survey paper [Kol10] for more details. One essential point of such varieties having a nice moduli theory is that they carry natural polarizations, namely, their canonical classes.

Another category of varieties carrying natural polarization is the class of Fano varieties, whose canonical classes are negative. However, such varieties behave quite differently with the canonically polarized varieties. For example, there exists a family of Fano manifolds, whose general fibers are isomorphic to a given Fano manifold, but the complex structure jumps at the special fibers (see e.g. [PP10, 2.3]). This means the functor of Fano manifolds in general is not separated. Nevertheless, even without knowing the uniqueness we can still ask generally whether a 'nice' degeneration exists.

Of course, this depends on the meaning of 'nice'. Here we are looking for degenerations with mild singularities whose canonical classes are still negative. Recall that a variety X is called a \mathbb{Q} -Fano variety if it only has klt singularities (see [KM98] for the meaning of the terminology) and $-K_X$ is ample. In particular, a \mathbb{Q} -Fano variety is normal. This class of varieties plays a central role in birational geometry. From many viewpoints, it has a similar behavior as Fano manifolds. Using the minimal model program with scaling to interpret the investigation in [Kol07a] and [HX09], we indeed obtain the following theorem on the existence of \mathbb{Q} -Fano degenerations.

Theorem 1. *Let $f^* : \mathcal{X}^* \rightarrow C^*$ be a projective morphism, \mathcal{X}^* with klt singularity, C the germ of a smooth curve, $p \in C$ a closed point and $C^* = C \setminus \{p\}$. If $-K_{\mathcal{X}^*}$ is f^* -ample and \mathcal{X}_t^* is a klt variety for any $t \in C^*$, then there is a finite dominating base change $d : C' \rightarrow C$, and a variety \mathcal{X} projective over C' such that the restriction of \mathcal{X} to the preimage $d^{-1}(C^*)$ is isomorphic to $\mathcal{X}^* \times_C C'$, $-K_{\mathcal{X}}$ is relatively ample over C' and every fiber \mathcal{X}_t is klt for any $t \in C'$. In particular, \mathcal{X}_t is normal.*

We call such degeneration a *special Fano degeneration*. The existence of a special Fano degeneration in general is somewhat surprising because for canonically polarized variety, the degenerations often only have semi-log canonical singularities. In particular, they are not always irreducible. We remark that in general special Fano degenerations are not unique. In fact, it is a recent result [HP10] which classifies all the special Fano degenerations of \mathbb{P}^2 .

However, even without the uniqueness, our construction of the degenerations behaves well for the calculation of the *Donaldson-Futaki intersection number*. We first need to define some terminologies.

Definition 1. 1. Let $f : (\mathcal{X}, \mathcal{L}) \rightarrow C$ be a flat projective morphism over a smooth curve, where \mathcal{X} is normal and \mathcal{L} is a f -ample \mathbb{Q} -line bundle. We call $(\mathcal{X}, \mathcal{L}) \rightarrow C$ a *polarized generic \mathbb{Q} -Fano family* if there exists a non-empty open set $C^* \subset C$, such that for any $t \in C^*$, \mathcal{X}_t is klt and $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{Q}, C^*} -rK_{\mathcal{X}^*}$ for some $r > 0$. In this case, we say C^* parametrizes non-degenerate fibers. If we can choose $C^* = C$, then we call $f : (\mathcal{X}, -K_{\mathcal{X}/C}) \rightarrow C$ a \mathbb{Q} -Fano family.
2. We say that $f : (\mathcal{X}, \mathcal{L}) \rightarrow C$ admits a G -action if G acts on $(\mathcal{X}, \mathcal{L})$ and C such that f is G -equivariant.

Given a polarized generic \mathbb{Q} -Fano family $(\mathcal{X}, \mathcal{L}) \rightarrow C$, there exists a maximal open set $C^* \subset C$ parametrizing non-degenerate fibers. Conversely, given a \mathbb{Q} -Fano family \mathcal{X}^* over C^* and $C^* \subset C$ where C is a smooth curve, using the properness of the Hilbert scheme, we easily see \mathcal{X}^* can be extended to be a generic \mathbb{Q} -Fano family over C .

Definition 2 (Donaldson-Futaki intersection number). *Let \mathcal{L} be a nef \mathbb{R} -line bundle on \mathcal{X} over a proper smooth curve C . We assume there is a non-empty open set $C^* \subset C$, such that for any $t \in C^*$, \mathcal{X}_t is klt and $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{R}, C^*} -rK_{\mathcal{X}^*}$ for some $r > 0$ which is ample over C^* . We define the Donaldson-Futaki intersection number (or DF intersection number) to be*

$$\text{DF}(\mathcal{X}/C, \mathcal{L}) = \frac{1}{2(n+1)r^n(-K_{\mathcal{X}_t})^n} \left((n+1)K_{\mathcal{X}/C} \cdot \mathcal{L}^n + \frac{n}{r}\mathcal{L}^{n+1} \right). \quad (1)$$

It is easy to see that if $\mathcal{L}_1 \sim_{\mathbb{R}, C} a\mathcal{L}_2$ for any $a > 0$, then

$$\text{DF}(\mathcal{X}/C, \mathcal{L}_1) = \text{DF}(\mathcal{X}/C, \mathcal{L}_2).$$

With this invariant, a more precise version of Theorem 1 is given by the following theorem.

Theorem 2. *Assume $(\mathcal{X}, \mathcal{L}) \rightarrow C$ is a polarized generic \mathbb{Q} -Fano family over a smooth proper curve C . Let $C^* \subset C$ parametrize non-degenerate fibers. Then there exists a finite morphism $\phi: C' \rightarrow C$ from a non-singular curve C' and a \mathbb{Q} -Fano family $(\mathcal{X}^s, \mathcal{L}^s := -rK_{\mathcal{X}^s}) \rightarrow C'$ such that there exists a birational map $\mathcal{X}^s \dashrightarrow \mathcal{X} \times_C C'$ which induces an isomorphism between the restrictions*

$$(\mathcal{X}^s, \mathcal{L}^s)|_{\phi^{-1}(C^*)} \cong (\mathcal{X}, \mathcal{L})|_{C^*} \times_C C'.$$

Moreover,

$$\text{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}) \leq \deg(\phi) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L})$$

and the equality holds if and only if $(\mathcal{X}, \mathcal{L}) \rightarrow C$ is a \mathbb{Q} -Fano family.

Furthermore, if $(\mathcal{X}, \mathcal{L}) \rightarrow C$ admits a \mathbb{G}_m -action, then we can assume the same is true for $(\mathcal{X}^s, \mathcal{L}^s) \rightarrow C'$ and the map $\mathcal{X}^s \dashrightarrow \mathcal{X} \times_C C'$ is \mathbb{G}_m -equivariant.

Our main application of Theorem 2, which is also the original motivation for this paper, is using it to study the test configurations and their Donaldson-Futaki invariants, especially to solve Tian's conjecture on special test configurations (see Corollary 2). We will discuss it in the next subsection.

1.2 K-stability from Kähler-Einstein problem

A fundamental problem in Kähler geometry is to determine whether there exists a Kähler-Einstein metric on a Fano manifold X , i.e., to find a Kähler metric ω_{KE} in the Kähler class $c_1(X)$ satisfying the equation:

$$\text{Ric}(\omega_{KE}) = \omega_{KE}.$$

This is a variational problem. Futaki [Fut83] found an important invariant as the obstruction to this problem. Then Mabuchi [Mab86] defined the K-energy functional by integrating this invariant. The minimizer of the K-energy is the Kähler-Einstein metric. Tian [Tia97] proved that there is a Kähler-Einstein metric if and only if the K-energy is proper on the space of all Kähler metrics in $c_1(X)$. So the problem is how to test the properness of the K-energy.

Tian developed a program to reduce this infinite dimensional problem to finite dimensional problems. More precisely, he proved in [Tia90] that the space of Kähler metrics in a fixed Kähler class can be approximated by a sequence of spaces consisting of Bergman metrics. The latter spaces are finite dimensional symmetric spaces. Tian ([Tia97]) then introduced the K-stability condition using the generalized Futaki invariant ([DT92]) for testing the properness of K-energy on these finitely dimensional spaces. Later Donaldson [Don02] reformulated it by defining the Futaki invariants algebraically (see (2)), which is now called the *Donaldson-Futaki invariant*. The following folklore conjecture is the guiding question in this area.

Conjecture 1 (Yau-Tian-Donaldson conjecture). *Let (X, L) be a polarized manifold. Then there is a constant scalar curvature Kähler metric in $c_1(L)$ if and only if (X, L) is K-polystable.*

In this paper, unless otherwise mentioned, we only consider the notion of K-stability for the Kähler-Einstein problem on Fano varieties (see [Tia10] for more detailed discussion for the Kähler-Einstein problem).

In the following we will recall the definition of K-stability. First we need to define the notion of *test configuration*.

Definition 3. 1. Let X be an n -dimensional \mathbb{Q} -Fano variety. Assume $-rK_X$ is Cartier. A test configuration of $(X, -rK_X)$ consists of

- a variety \mathcal{X}^{tc} with a \mathbb{G}_m -action,
- a \mathbb{G}_m -equivariant ample line bundle $\mathcal{L}^{tc} \rightarrow \mathcal{X}^{tc}$,
- a flat \mathbb{G}_m -equivariant map $\pi : (\mathcal{X}^{tc}, \mathcal{L}^{tc}) \rightarrow \mathbb{A}^1$, where \mathbb{G}_m acts on \mathbb{A}^1 by multiplication in the standard way $(t, a) \rightarrow ta$,

such that for any $t \neq 0$, $(\mathcal{X}_t^{tc} = \pi^{-1}(t), \mathcal{L}_t^{tc}|_{\mathcal{X}_t^{tc}})$ is isomorphic to $(X, -rK_X)$.

2. If \mathcal{L}^{tc} is only a \mathbb{Q} -Cartier divisor on \mathcal{X}^{tc} such that for an integer $m \geq 1$, $(\mathcal{X}^{tc}, m\mathcal{L}^{tc})$ yields a test configuration of $(X, -mrK_X)$. We call $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ a \mathbb{Q} -test configuration of $(X, -rK_X)$.

Remark 1. In the following, by the abuse of notation, if we do not want to specify the exponent r , we will call $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ a test configuration for both cases in the above definition. Therefore by rescaling the polarization, any test configuration can be considered obtained from a \mathbb{Q} -test configuration of $(X, -K_X)$.

Similarly to the notion of \mathbb{Q} -Fano family, we have the following definition.

Definition 4. A \mathbb{Q} -test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ of $(X, -rK_X)$ is called a *special test configuration* if $\mathcal{L}^{tc} \sim_{\mathbb{Q}} -rK_{\mathcal{X}^{tc}}$ and \mathcal{X}_0^{tc} is a \mathbb{Q} -Fano variety.

For any test configuration, we can define the *Donaldson-Futaki invariant*. First by the Riemann-Roch theorem, for sufficiently divisible k ,

$$d_k = \dim H^0(X, \mathcal{O}_X(-rkK_X)) = a_0k^n + a_1k^{n-1} + O(k^{n-2})$$

for some rational numbers a_0 and a_1 . Let $(\mathcal{X}_0^{tc}, \mathcal{L}_0^{tc})$ be the restriction of $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ over $\{0\}$. Since \mathbb{G}_m acts on $(\mathcal{X}_0^{tc}, \mathcal{L}_0^{tc \otimes k})$, \mathbb{G}_m also acts on $H^0(\mathcal{X}_0^{tc}, \mathcal{L}_0^{tc \otimes k})$. We denote its total \mathbb{G}_m -weight by w_k . By the equivariant Riemann-Roch Theorem,

$$w_k = b_0k^{n+1} + b_1k^n + O(k^{n-1}).$$

So we can expand

$$\frac{w_k}{kd_k} = F_0 + F_1k^{-1} + O(k^{-2}).$$

Definition 5 ([Don02]). The (normalized) Donaldson-Futaki invariant (DF-invariant) of the test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ is defined to be

$$\text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = -\frac{F_1}{a_0} = \frac{a_1b_0 - a_0b_1}{a_0^2} \quad (2)$$

With the normalization in (2), we easily see $\text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = \text{DF}(\mathcal{X}^{tc}, a\mathcal{L}^{tc})$ for any $a \in \mathbb{Q}_{>0}$.

Remark 2. We have the following remarks about Donaldson-Futaki invariant.

1. From the differential geometry side, Ding and Tian [DT92] defined the generalized Futaki invariants by extending the differential geometric formula of Futaki [Fut83] from smooth manifolds to normal varieties. On a normal \mathbb{Q} -Fano variety, the differential geometric definition coincides with the above algebraic definition. This was proved by Donaldson [Don02] in the smooth case. The calculation via equivariant forms in [Don02] is also valid in the normal case, because the codimension of singularities on a normal variety is at least two.

2. In [PT06] and [PT09], Paul and Tian proved the Donaldson-Futaki invariant is the same as the total \mathbb{G}_m -weight of the CM line bundle, which was first introduced by Tian to give a GIT formulation of K-stability. See [Tia97] for details.

As we show in Section 3, if $C = \mathbb{P}^1$ and $(\bar{\mathcal{X}}^{tc}, \bar{\mathcal{L}}^{tc}) \rightarrow \mathbb{P}^1$ is the compactification of a \mathbb{Q} -test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \rightarrow \mathbb{A}^1$ as in Section 3.2 by simply adding a ‘trivial fiber’ over the point $\{\infty\} \in \mathbb{P}^1$, then we have the equality

$$\mathrm{DF}(\bar{\mathcal{X}}^{tc}/\mathbb{P}^1, \bar{\mathcal{L}}^{tc}) = \mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}),$$

which explains the origin of our terminology.

If we apply our Theorem 2 to this case, it specializes to the following result.

Corollary 1. *Let X be a \mathbb{Q} -Fano variety and $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ a test configuration of $(X, -rK_X)$. We can construct a special test configuration $(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}})$ and a positive integer m , such that*

$$m\mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \geq \mathrm{DF}(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}}).$$

Furthermore, if we assume \mathcal{X}^{tc} is normal, then the equality holds only when $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ itself is a special test configuration.

This corollary will be applied to study K-stability, which we define in the below.

Definition 6. *Let X be a \mathbb{Q} -Fano variety.*

1. X is called *K-semistable* if for any \mathbb{Q} -test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ of $(X, -rK_X)$ with $r > 0$, we have $\mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \geq 0$.
2. X is called *K-stable* (resp. *K-polystable*) if for any normal \mathbb{Q} -test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ of $(X, -rK_X)$ with $r > 0$, we have $\mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \geq 0$, and the equality holds only if $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ is trivial (resp. $\mathcal{X}^{tc} \cong X \times \mathbb{A}^1$).

Remark 3. *We have the following remarks for the definitions of K-stability.*

1. Though the notions of K-stability can be stated for a general singular variety X with $-K_X$ being \mathbb{Q} -Cartier and ample, in [Oda08], Odaka shows that for such a variety, if it is K-semistable, it can only have klt singularities.
2. In the definitions of K-polystability and K-stability, for the triviality of the test configuration with Donaldson-Futaki invariant 0, we require the test configuration to be normal. This is slightly different with the original definition. However, we believe this should be the right one. For more details, see Subsection 3.1. It is a consequence of [RT07, 5.2] that we only need to consider normal test configurations for K-semistability, too.

In the work [Tia97] where Tian gave the original definition of the K-stability by using the analytic language, he only considered test configurations with normal central fibers. Later he conjectured that, at least for Fano manifolds, even with Donaldson’s definition, one still only needs to consider those test configurations with normal central fibers. This is motivated by compactness results for Kähler-Einstein manifolds (See [CCT02]).

As an immediate consequence of Corollary 1, we verify Tian’s conjecture. In fact, it suffices to consider an even smaller class of test configurations, namely, the test configurations whose central fibers are \mathbb{Q} -Fano.

Corollary 2 (Tian’s conjecture). *Assume X is a \mathbb{Q} -Fano variety. If X is destabilized by a test configuration, then X is indeed destabilized by a special test configuration. More precisely, the following two statements are true.*

1. (*K-semistability*) If $(X, -rK_X)$ is not *K-semi-stable*, then there exists a special test configuration $(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}})$ with a negative Futaki invariant $\text{DF}(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}}) < 0$.
2. (*K-polystability*) Let X be a *K-semistable* variety. If $(X, -rK_X)$ is not *K-polystable*, then there exists a special test configuration $(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}})$ with Donaldson-Futaki invariant 0 such that \mathcal{X}^{st} is not isomorphic to $X \times \mathbb{A}^1$.

In fact, these results should also be considered as an evidence that the Gromov-Hausdorff limit of a family of Fano manifolds with KE metrics, which people conjecture to admit a complex analytic structure see [Tia10]), should be a \mathbb{Q} -Fano variety with KE metric. Even more strongly, it suggests that the moduli space of \mathbb{Q} -Fano varieties admitting KE metric should form a compact moduli.

1.3 Outline of the proof

In this subsection, we explain our strategy. To connect it to the earlier work and to simplify the notations a little bit, we only discuss the case that when our family is a test configuration.

The main idea of showing Theorem 2 is to modify a given test configuration, and then to use the intersection formula to analyze the variation of Donaldson-Futaki invariants under modifications of the test configurations. In [RT07] and [ALV11], the authors have also studied how to simplify a given test configuration. In particular, by using the (equivariant) semistable reduction theorem from [KKMS73], it was shown (cf. [ALV11]) that any *K-unstable* polarized variety (X, L) can be destabilized by a test configuration whose central fibre \mathcal{X}_0^{tc} is (reduced) simple normal crossing. In our paper, we will apply the machinery of minimal model program with scaling to modify this semistable test configuration. As far as we know, the idea of applying the modern birational geometry to study *K-stability* algebraically is first contained in Odaka's work (see [Oda08]). Here we carry out a more refined study of the change of Donaldson-Futaki invariants under various birational modifications coming from MMP.

Our first observation is that the DF intersection numbers of the polarizations always *decrease* along the direction of the canonical class of the test configuration. Of course, when we deform the polarization along the canonical class for enough long time, we may hit the boundary of ample cone. Then MMP allows us to change the model to enable us to continue the deformation. So as long as it is before the pseudo-effective threshold, we still get polarizations but on some new models. In fact in birational geometry, this process is precisely the minimal model program with scaling which is a central theme in many recent work, e.g., [BCHM10].

On the other hand, we can show if we run an MMP from the semi-stable model with the scaling of a suitably perturbed polarization, at the pseudo-effective threshold point we contract all but one components and end with a special test configuration. Since by our observation, the Donaldson-Futaki invariants decrease along the sequence of MMP with scaling, this indeed finishes the proof of the *K-semistable* case.

However, to prove the *K-stable* case, we have to eliminate the perturbation term appearing when we perturb the polarization. This is more involving and therefore we have to take a more delicate process. In fact we have to divide the modification process into three steps and the strategy also works for a general base curve.

Step 1. We first apply semi-stable reduction and then run a relative MMP of this semi-stable family over \mathcal{X} . Then we achieve a model \mathcal{X}^{lc} which is the *log canonical modification* of the base change of $(\mathcal{X}, \mathcal{X}_0)$. We define the polarization \mathcal{L}^{lc} on \mathcal{X}^{lc} to be the sum of the pull back of the original polarization and a small positive multiple of $K_{\mathcal{X}^{lc}}$, i.e., the new polarization on $K_{\mathcal{X}^{lc}}$ is obtained by deforming the original one along the direction of the canonical class. Thus by our observation, we can check that the DF-intersection numbers decrease. We note that the idea of running the MMP by passing through the log canonical modification is inspired by Odaka's work (see [Oda09]).

Theorem 3. *Let $(\mathcal{X}, \mathcal{L}) \rightarrow C$ be a polarized generic \mathbb{Q} -Fano family over a proper smooth curve, with $C^* \subset C$ parametrizing the non-degenerate fibers. Then there is a finite morphism $\phi : C' \rightarrow C$ and a polarized generic \mathbb{Q} -Fano family $(\mathcal{X}^{lc}, \mathcal{L}^{lc}) \rightarrow C'$ with the following properties:*

1. *There is a birational morphism $\mathcal{X}^{lc} \rightarrow \mathcal{X} \times_C C'$, which is isomorphic over $\phi^{-1}(C^*)$.*
2. *For every $t \in C'$, $(\mathcal{X}^{lc}, \mathcal{X}_t^{lc})$ is log canonical.*
3. *There is an inequality*

$$\mathrm{DF}(\mathcal{X}^{lc}/C', \mathcal{L}^{lc}) \leq \deg(\phi) \cdot \mathrm{DF}(\mathcal{X}/C, \mathcal{L}).$$

Moreover, the equality holds if and only if $(\mathcal{X}, \mathcal{X}_t)$ is log canonical for every $t \in C$, in which case \mathcal{X}^{lc} is isomorphic to $\mathcal{X} \times_C C'$.

Step 2. Next, we will run the minimal model program with scaling. Replacing \mathcal{L}^{lc} by its multiple, we can assume that $\mathcal{H}^{lc} = \mathcal{L}^{lc} - K_{\mathcal{X}^{lc}}$ is ample. Thus we can run $K_{\mathcal{X}^{lc}}$ -MMP with scaling of \mathcal{H}^{lc} over C . It follows from [BCHM10] that the sequence of $K_{\mathcal{X}^{lc}}$ -MMP with scaling of \mathcal{H}^{lc} over C will decrease the scaling factor until the pseudo-effective threshold is reached. We denote its anti-canonical model by \mathcal{X}^{an} . Since this is again deforming the polarization along the direction of the canonical class, we can also check the DF intersection numbers are continuously decreasing when we scale down the scaling factor. So we have the following theorem.

Theorem 4. *Let $(\mathcal{X}^{lc}, \mathcal{L}^{lc}) \rightarrow C$ be a polarized generic \mathbb{Q} -Fano family over a proper smooth curve, with $C^* \subset C$ parametrizing the non-degenerate fibers. We assume $(\mathcal{X}^{lc}, \mathcal{X}_t^{lc})$ is log canonical for any $t \in C$, then there is a polarized generic \mathbb{Q} -Fano family $(\mathcal{X}^{an}, \mathcal{L}^{an}) \rightarrow C$ which is isomorphic to $(\mathcal{X}^{lc}, \mathcal{L}^{lc})$ over C^* , such that $-rK_{\mathcal{X}^{an}} \sim_{\mathbb{Q}, C} \mathcal{L}^{an}$, $(\mathcal{X}^{an}, \mathcal{X}_t^{an})$ is log canonical for any $t \in C$ and*

$$\mathrm{DF}(\mathcal{X}^{an}/C, \mathcal{L}^{an}) \leq \mathrm{DF}(\mathcal{X}^{lc}/C, \mathcal{L}^{lc}),$$

with the equality holding if and only if $(\mathcal{X}^{an}, \mathcal{L}^{an}) \cong (\mathcal{X}^{lc}, \mathcal{L}^{lc})$.

Step 3. At this point, running MMP of \mathcal{X}^{an} with scaling of $\mathcal{L}^{an} \sim_{\mathbb{Q}, C} -rK_{\mathcal{X}^{an}}$ will not produce any new model but we still need to construct a model \mathcal{X}^s such that $(\mathcal{X}^s, \mathcal{X}_t^s)$ is plt. In stead of running MMP, now we apply the technique of extending \mathbb{Q} -Fano varieties. So after possibly a base change we obtain a \mathbb{Q} -Fano family \mathcal{X}^s which is isomorphic to \mathcal{X}^* over C^* . Furthermore, from our construction of \mathbb{Q} -Fano extension, we know that \mathcal{X}_t^s is a divisor whose discrepancy with respect to \mathcal{X}^{an} is 0.

Thus we can extract \mathcal{X}_t^s on \mathcal{X}^{an} to get a model \mathcal{X}' such that $-K_{\mathcal{X}'}$ is relatively big and nef, and $\mathcal{X}' \dashrightarrow \mathcal{X}^s$ is a birational contraction. Since over those models whose anti-canonical class is proportional to the polarization over the base, the DF intersection number is just a linear function on the volume of the anti-canonical class whose linear term has a negative coefficient. Thus we easily see it decreases from \mathcal{X}' to \mathcal{X}^s . This can be also interpreted as the fact that DF intersection numbers decrease when we deform the polarization along the extra components.

Theorem 5. *Let $(\mathcal{X}^{an}, \mathcal{L}^{an}) \rightarrow C$ be a polarized generic \mathbb{Q} -Fano family, where $C^* \subset C$ parametrizes non-degenerate fibers. We assume $-rK_{\mathcal{X}^{an}} \sim_{\mathbb{Q}, C} \mathcal{L}^{an}$. Then after a base change $\phi : C' \rightarrow C$, there is a \mathbb{Q} -Fano family, $(\mathcal{X}^s, -rK_{\mathcal{X}^s}) \rightarrow C'$ which is isomorphic to $(\mathcal{X}^{an}, \mathcal{L}^{an}) \times_C C'$ over $\phi^{-1}(C^*)$, such that*

$$\mathrm{DF}(\mathcal{X}^s/C', -rK_{\mathcal{X}^s}) \leq \deg(\phi) \cdot \mathrm{DF}(\mathcal{X}^{an}/C, -rK_{\mathcal{X}^{an}}),$$

and the equality holds if and only if $(\mathcal{X}^s, -rK_{\mathcal{X}^s})$ is a \mathbb{Q} -Fano family.

We want to emphasize it is because that we always deform the polarization along a ‘carefully chosen’ direction that the DF intersection numbers decrease. And all the theorems 3, 4 and 5 have an equivariant version at least for the group \mathbb{G}_m , which we omit.

Finally, we outline the organization of this paper. In Section 2, we review the results which we need from birational geometry and MMP. At the end we prove Theorem 7 which is a more precise form of Theorem 1. In Section 3, we give an amendment of the K-stability condition. Then we study the Donaldson-Futaki invariant and prove it can be written as a DF-intersection number. In Section 4, 5 and 6, we prove Theorem 3, 4 and 5 respectively. At the last section, we put them together to obtain Theorem 2 the main theorem.

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2 MMP and birational models

In this section, we give a few definitions as well as briefly introduce some preliminary results, especially the results in the minimal model program. Then we prove Theorem 7 which is a more precise form than Theorem 1. Later this form is needed for the purpose of calculating the variance of Donaldson-Futaki intersection numbers.

2.1 Notation and Conventions

We follow [KM98] for the standard terminologies of birational geometry. In particular, see [KM98, 0.4] for the definitions of general concepts and [KM98, 2.34, 2.37] for the definitions of *klt*, *lc* and *dlt* singularities.

A smooth variety Y which is flat over a smooth curve C is called a *semi-stable family* if for any $t \in C$, (Y, Y_t) is simple normal crossing.

Let $f : (X, \Delta) \rightarrow S$ be a log canonical pair projective over S . For any \mathbb{Q} -divisor D on X , we denote $\oplus_m f_*([mD])$ by $R(X/S, D)$. Assume X is proper over S . A \mathbb{Q} -Cartier \mathbb{Q} -divisor L on X is called *relatively semi-ample* if for sufficiently divisible $m > 0$, $f^* f_* \mathcal{O}_X(mL) \rightarrow \mathcal{O}_X(mL)$ is surjective. Assume $K_X + \Delta$ is big and semi-ample over S , we call

$$Y = \text{Proj } R(X/S, K_X + \Delta)$$

the *relative log canonical model* of (X, Δ) over S . It is easy to see that Y is the image of the linear system $|m(K_X + \Delta)|$ for sufficiently divisible $m > 0$. We say X^m is a *good minimal model* of (X, Δ) over S if $h : X \dashrightarrow X^m$ is a minimal model of (X, Δ) over S and $K_{X^m} + h_* \Delta$ is relatively semi-ample.

Let (X, Δ) be a normal pair, i.e., X is a normal variety and $\Delta = \sum a_i \Delta_i$ is a \mathbb{Q} -divisor with distinct prime divisors Δ_i and rational numbers a_i . Assume $0 \leq a_i \leq 1$. We say that a birational projective morphism $f : Y \rightarrow (X, \Delta)$ gives the *log canonical modification* of (X, Δ) if with the divisor $\Delta_Y = f_*^{-1}(\Delta_X) + E^{lc}$ on Y , the pair (Y, Δ_Y) satisfies

- (1) (Y, Δ_Y) is log canonical, and
- (2) $K_Y + \Delta_Y$ is ample over X .

Here E^{lc} denotes the sum of f -exceptional prime divisors with coefficients 1.

Let $X \rightarrow Y$ be a dominant morphism between normal varieties. A prime divisor E of X is called *horizontal* if E dominates Y , otherwise it is called *vertical*. Given any divisor E , we can uniquely write $E = E^v + E^h$ where the horizontal part E^h consists of all the horizontal components and the vertical part E^v consists of all the vertical components.

For a model \bullet (e.g., \mathcal{X}) over C and $0 \in C$ a point, we denote by \bullet_0 its fiber over 0.

2.2 Equivariant Semi-stable reduction

Lemma 1. *Let $f : \mathcal{X} \rightarrow \mathbb{A}^1$ be a dominant morphism from a normal variety with a \mathbb{G}_m -action such that f is \mathbb{G}_m -equivariant. Then there exists a base change $z^m : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ and a semistable family \mathcal{Y} over \mathbb{A}^1 with a morphism $\pi : \mathcal{Y} \rightarrow \mathcal{X} \times_C C'$ which is a log resolution of $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$ where $\tilde{\mathcal{X}}$ is the normalization of $\mathcal{X} \times_C C'$.*

A similar statement but only for resolution appears in [ALV11].

Proof. Without the \mathbb{G}_m -action, this is just [KM98, 7.17]. We follow the strategy there. First, we perform the blow up of $(\mathcal{X}, \mathcal{X}_0)$ \mathbb{G}_m -equivariantly to get an equivariant log resolution \mathcal{Y}^* . This is always possible by the theorem of equivariant resolution of singularities (see [Kol07b]). Then we can take a base change $z^m : \mathbb{A}^1 \rightarrow \mathbb{A}^1$, such that the normalization $\tilde{\mathcal{Y}}^*$ of $\mathcal{Y}^* \times_C C'$ has a reduced fiber over each point of \mathbb{A}^1 . Then it follows from [KKMS73], that we can take a sequence of toroidal blow-ups of $\tilde{\mathcal{Y}}^*$ such that it is a log resolution \mathcal{Y} of $(\tilde{\mathcal{Y}}^*, \tilde{\mathcal{Y}}_0^*)$ and still only has reduced fibers. Since \mathbb{G}_m acts on the toroidal structure trivially, then we know $\tilde{\mathcal{Y}}^*$ automatically admits a \mathbb{G}_m -action which makes $\mathcal{Y} \rightarrow \mathbb{A}^1$ a \mathbb{G}_m -equivariant morphism. \square

2.3 MMP with scaling

In this subsection, we briefly introduce the concept of MMP with scaling and summarize the results which we will need later.

Let $(X, \Delta)/S$ be a klt pair, which is projective over S . Let H be an ample class on X . Let $\lambda \geq 0$ be a positive number such that $K_X + \Delta + \lambda H$ is nef over S . For instance, we can take $\lambda \gg 0$. Denote $(X^0, \Delta^0) := (X, \Delta)$ and $\lambda_0 = \lambda$.

Suppose we have defined a klt pair (X^i, Δ^i) which is projective over S , a \mathbb{Q} -divisor H^i on X^i , and a positive number λ_i such that $K_{X^i} + \Delta^i + \lambda_i H^i$ is nef over S . We first define

$$\lambda_{i+1} = \min\{\lambda | K_{X^i} + \Delta^i + \lambda H^i \text{ is nef over } S\} \quad (3)$$

If $\lambda_{i+1} = 0$, then we stop. Otherwise, suppose we can choose a $(K_{X^i} + \Delta^i)$ -negative extremal ray $R \subset \text{NE}(X^i/S)$, with $R \cdot (K_{X^i} + \Delta^i + \lambda_{i+1} H^i) = 0$. Assume the extremal contraction induced by R is birational. Then we let X^{i+1} be the variety obtained by the $(K_{X^i} + \Delta^i)$ -flip or divisorial contraction over S with respect to the curve class R (cf. [KM98, Section 3.7]), and Δ^{i+1} (resp. H^{i+1}) the push-forward of Δ^i (resp. H^i) to X^{i+1} . It is obvious that $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_i \geq \dots$. We call this sequence

$$(X^0, \Delta^0) \dashrightarrow (X^1, \Delta^1) \dashrightarrow \dots \dashrightarrow (X^i, \Delta^i) \dashrightarrow \dots,$$

a sequence of $(K_X + \Delta)$ -MMP with scaling of H , λ the scaling factor and H the scaling divisor class (or the scaling divisor for abbreviation). We note that this special MMP sequence has been extensively studied since [BCHM10].

From the definition, we know that for any $t \in [\lambda_{i+1}, \lambda_i]$, $K_{X^i} + \Delta^i + tH^i$ is nef over S . Moreover, if $t \in [\lambda_{i+1}, \lambda_i]$, then X^i is a relatively minimal model of $(X, \Delta + tH)$ over S because for $j \leq i$, the step $X^{j-1} \dashrightarrow X^j$ of $(K_X + \Delta)$ -MMP is also a step for $(K_X + \Delta + tH)$ -MMP.

We need the following results.

Theorem 6. *With the above notations. Let (X, Δ) be a klt pair which is projective over S . There exists a finite number i such that*

1. *if Δ is big and $K_X + \Delta$ is pseudo-effective over S , then after i steps, $\lambda_i = 0$ and $K_{X^i} + \Delta^i$ is semi-ample over S ;*
2. *if $K_X + \Delta$ is not pseudo-effective, then after i steps, λ_i is equal to the pseudo-effective threshold (see [BCHM10, 1.1.6] for the definition) of $K_X + \Delta$ with respect to H , which is a rational number. Furthermore, if we let i be the smallest index such that λ_i is equal to the pseudo-effective threshold, then X^{i-1} is a good minimal model of $(X, \Delta + \lambda_i H)$ over C .*

Proof. See [BCHM10, 1.1.7 and 1.4.2]. □

Let G be a connected algebraic group, e.g., $G = \mathbb{G}_m$. Let $(\mathcal{X}, \mathcal{L}) \rightarrow C$ be a generic \mathbb{Q} -Fano family admitting a G -action and $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ an equivariant resolution. In the following, we will consider two types of scaling divisor H when we run MMP:

1. on \mathcal{X} , we will simply let the scaling divisor be the class of the polarization \mathcal{L} , or
2. on \mathcal{Y} , the scaling divisor will be $\pi^* \mathcal{L} + E$ for some divisor E supported either on $\text{Ex}(\pi)$ or on an invariant fiber \mathcal{Y}_0 .

Since G is connected and π is G -equivariant, any component of $\text{Ex}(\pi)$ is G -invariant. Thus we see in the above two cases, the scaling divisors are G -invariant.

In general, if a connected algebraic group G acts $(X, \Delta)/S$ such that the scaling divisor H is G -invariant, then when we run a relative $(K_X + \Delta)$ -MMP over S with the scaling of H , each step is indeed automatically G -equivariant. In fact, assuming this is true after the i -th step, since G is connected, then $\text{NE}(X^i)^G = \text{NE}(X^i)$ (see the proof of [Adr01, 1.5]). Hence the contraction is G -equivariant. As the flip is a Proj of a G -equivariant algebra, it also admits a G -action. Obviously, all the maps between the appearing models are G -equivariant. More generally, the above discussion still holds if instead of G acts Δ , we only assume that the \mathbb{Q} -linear equivalent class of Δ is fixed by G .

Remark 4 (Equivariant models). *We have seen resolution of singularities and semistable reduction can be obtained G -equivariantly (see Lemma 1). And the above discussion shows that the MMP with scaling is also G -equivariant if we assume the starting pair and the scaling divisor are equivariant.*

In this paper, these are the three types of modifications for a given pair (X, Δ) . Therefore, if we start with a generic \mathbb{Q} -Fano family which is G -equivariant, then all the models obtained are also G -equivariant. To avoid the repetition, in the following, we will not mention the G -action explicitly any more.

Proposition 1. *Let (Y, Δ_Y) be a klt pair projective over a smooth curve C with a relative ample class L . We assume that we can write $K_Y + \Delta_Y + L \sim_{\mathbb{Q}, C} E = E^h + E^v \geq 0$ such that the horizontal part E^h is exceptional for a birational morphism $Y \rightarrow X$, and the vertical part E^v can be written as $\sum a_i E_i^v$ where $a_i > 0$, and $\text{Supp}(\sum E_i^v)$ does not contain any fiber. Then the $(K_Y + \Delta_Y + L)$ -MMP with scaling of L will yield a model Y^i such that $K_{Y^i} + \Delta_{Y^i} + L^i \sim_{\mathbb{Q}} 0$ where Δ_{Y^i} and L^i are the push forward of Δ_Y and L on Y^i . Furthermore, the divisors contracted by $Y \dashrightarrow Y^i$ are precisely $\text{Supp}(E)$, and L^i is big and nef on Y^i .*

Proof. From the assumption, we know that the pseudo-effective threshold of $(Y, \Delta_Y)/C$ with respect to L is 1. Then by Theorem 6(2), this sequence of MMP yields a good minimal model Y^i of $K_Y + \Delta_Y + L$ over C . Since $K_{Y^i} + \Delta_{Y^i} + L^i$ is semi-ample, the map $Y \dashrightarrow Y^i$ contracts precisely the divisorial component in $\mathbf{B}(K_Y + \Delta_Y + L/C)$ which is $\text{Supp}(E)$. In fact, it is easy to see this for the components in E^h since they are exceptional for a birational morphism. For E^v , by our assumption, it is of the insufficient fiber type (cf. [Lai11, 2.9]), so by [Lai11, 2.10] it is contained in

$$\mathbf{B}_-(K_Y + \Delta_Y + L/C) \subset \mathbf{B}(K_Y + \Delta_Y + L/C).$$

From the definition of MMP with scaling, we see that for any $t \in [\lambda_{i+1}, \lambda_i]$, $K_{Y^i} + \Delta_{Y^i} + tL^i$ is nef. Since $K_{Y^i} + \Delta_{Y^i} + L^i \sim_{\mathbb{Q}, C} 0$ and by our assumption $\lambda_i > \lambda_{i+1} = 1$, then L^i is nef over C . \square

2.4 Log canonical modification and \mathbb{Q} -Fano extension

Let $f^* : \mathcal{X}^* \rightarrow C^*$ be a flat projective morphism, \mathcal{X}^* a klt variety, C^* the germ of a smooth curve. Let C be a smooth curve such that $C^* = C \setminus \{0\}$. Let \mathcal{X} be a normal compactification of \mathcal{X}^* which is projective over C such that $\mathcal{X}^* = \mathcal{X} \times_C C^*$. We first show a general result of the existence of the log canonical modification for the variety fibered over a curve. In fact, the log canonical modification is known to exist under more general assumptions (see e.g., [OX11]). Here we just give a proof of the case that we need for the reader's convenience.

Proposition 2. *With the above notations. Assume \mathcal{X} admits a log resolution \mathcal{Y} , such that \mathcal{Y}_0 is reduced simple normal crossing. Then the log canonical modification $\mathcal{X}^{lc} \rightarrow (\mathcal{X}, \mathcal{X}_0)$ exists and satisfies $(\mathcal{X}^{lc}, \mathcal{X}_0^{lc})$ is log canonical.*

Proof. Let $\pi : \mathcal{Y} \rightarrow (\mathcal{X}, \mathcal{X}_0)$ be a log resolution. If we write B to be the reduced divisor $\text{Ex}(\pi)$, it is well-known that it suffices to show that $(\mathcal{Y}, B + \pi_*^{-1}\mathcal{X}_0)$ has a relative log canonical model over \mathcal{X} (see [OX11, Lemma 2.2]). Write

$$\pi^*(K_{\mathcal{X}^*}) + F^* = K_{\mathcal{Y}^*} + E^*,$$

where F^*, E^* are effective and without common components. Let E be the closure of E^* in \mathcal{Y} . Now we consider the pair $(\mathcal{Y}, E + \delta G)$ where G is the sum of the π -exceptional divisors whose centers are over C^* and $0 < \delta \ll 1$ such that $(\mathcal{Y}, E + \delta G)$ is klt. Then it follows from [BCHM10] that $R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G)$ is a finitely generated $\mathcal{O}_{\mathcal{X}}$ -algebra. By taking Proj , we obtain a model $\phi : \mathcal{Y} \dashrightarrow \mathcal{X}^{lc}$ over \mathcal{X} . The model \mathcal{X}^{lc} is isomorphic to $\text{Proj} R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0)$ since \mathcal{Y}_0 is the pull back of a divisor from \mathcal{X} .

Because $D = B + \pi_*^{-1}\mathcal{X}_0 - E - \delta G - \mathcal{Y}_0 \geq 0$ is an effective divisor, we know that

$$K_{\mathcal{Y}} + B + \pi_*^{-1}\mathcal{X}_0 - \phi^*\phi_*(K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0) \geq K_{\mathcal{Y}} + B + \pi_*^{-1}\mathcal{X}_0 - (K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0) \geq 0.$$

Since ϕ contracts G which is the same as $\text{Supp}(D)$, we easily see this implies that there is an isomorphism

$$R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + \pi_*^{-1}\mathcal{X}_0 + B) \cong R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0).$$

Hence we see \mathcal{X}^{lc} is indeed the log canonical modification of $(\mathcal{X}, \mathcal{X}_0)$ and $(\mathcal{X}^{lc}, \mathcal{X}_0^{lc})$ is log canonical as $\phi_*(E + \delta G + \mathcal{Y}_0) = \mathcal{X}_0^{lc}$. \square

Next we study degenerations of Fano varieties.

Example 1 (Degenerations of cubic surfaces). *Let us consider two families whose generic fibers are cubic surfaces:*

$$\mathcal{X} = (tf_3(x, y, z, w) + xyz = 0) \quad \text{and} \quad \mathcal{Y} = (g_3(x, y, z) + tw^3 = 0)$$

contained in $\mathbb{P}(x, y, z, w) \times k[t]$, where f_3 (resp. g_3) is a general degree 3 homogeneous polynomial of x, y, z and w (resp. x, y and z). Projecting to the second factor, \mathcal{X} and \mathcal{Y} are families of cubic surfaces over \mathbb{A}^1 whose general fibers are smooth.

Now we modify \mathcal{X} in the following way: First we blow up the point

$$(0, 0, 0, 1) \in \mathcal{X}_0 = \sum_{i=1}^3 E_i = (xyz = 0) \subset \mathbb{P}^3$$

to get \mathcal{X}' and we denote the exceptional divisor by $E \cong \mathbb{P}^2$. The fiber \mathcal{X}'_0 has multiplicity 3 along E . Each birational transform of E_i is isomorphic to the \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over \mathbb{P}^1 . Next we take a degree 3 base change $k[t] \rightarrow k[t_1]$ which sends t to t_1^3 . Let \mathcal{X} be the normalization of $\mathcal{X}' \times_{k[t]} k[t_1]$. The pre-image S_1 of E in \mathcal{X} is the degree 3 cover branched over the intersection of E and the birational transform of $\sum_{i=1}^3 E_i$, which is isomorphic to $(xyz = 0) \subset \mathbb{P}(x, y, z) \cong E$. Hence S_1 is a cubic surface with three A_2 singularities. We can contract the preimage of the birational transform of $\sum_{i=1}^3 E_i$ and get a model \mathcal{X}^s . It is easy to see that $\mathcal{X}_0^s \cong S_1$.

We modify \mathcal{Y} in a similar way: We blow up the point $(0, 0, 0, 1) \in \mathcal{Y}_0 \subset \mathbb{P}^3$. Note the special fiber $\mathcal{Y}_0 \subset \mathbb{P}^3$ is a cone over the elliptic curve and $(0, 0, 0, 1)$ is the singularity. Denote by \mathcal{Y}' the new model, $E \cong \mathbb{P}^2$ the exceptional divisor and F the birational transform of \mathcal{Y}_0 which is isomorphic to a \mathbb{P}^1 -bundle over an elliptic curve. The intersection $E \cap F$, which is isomorphic to the cubic curve $(g_3) = 0 \subset \mathbb{P}^2 \cong E$, is a section of this \mathbb{P}^1 -bundle. The fiber \mathcal{Y}'_0 has multiplicity 3 along E . Taking the degree 3 base change $k[t] \rightarrow k[t_1]$ which sends t to t_1^3 and letting $\tilde{\mathcal{Y}}$ be the normalization of $\mathcal{Y}' \times_{k[t]} k[t_1]$, the pre-image S_2 of E in $\tilde{\mathcal{Y}}$ is the degree 3 cover branched over the intersection of E and F . Hence S_2 is a smooth cubic surface. We can contract the preimage of F and get a model \mathcal{Y}^s with $\mathcal{Y}_0^s \cong S_2$.

We remark that the second model \mathcal{Y} is a *standard model* in the sense of [Cor96]. Unlike Corti's theory, we allow base change. Hence not surprisingly, we can improve the singularity of the special fiber of even a standard model.

This construction can be indeed generalized to arbitrary polarized generic \mathbb{Q} -Fano families in the following sense.

Theorem 7 (\mathbb{Q} -Fano extension). *Let $f : \mathcal{X} \rightarrow C$ be a projective morphism over a smooth curve C . Assume $\mathcal{X}^* = \mathcal{X} \times_C C^*$ is a klt variety, where $C^* = C \setminus \{0\}$. We assume that there exists an ample \mathbb{Q} -divisor \mathcal{L} on \mathcal{X} such that $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{Q}, C^*} -K_{\mathcal{X}^*}$ and \mathcal{X}_t^* is a \mathbb{Q} -Fano variety for any $t \in C^*$.*

1. *There is a finite morphism $\phi : C' \rightarrow C$ and a klt variety \mathcal{X}^s projective over C' such that the restriction of \mathcal{X}^s to the preimage $\phi^{-1}(C^*)$ is isomorphic to $\mathcal{X}^* \times_C C'$ and all fibers \mathcal{X}_t^s is \mathbb{Q} -Fano variety. In particular, \mathcal{X}_t^s is normal.*
2. *Moreover, if we assume $f^{an} : \mathcal{X} = \mathcal{X}^{an} \rightarrow C$ is a normal compactification such that $\mathcal{L}^{an} = -K_{\mathcal{X}^{an}}$ is anti-ample and for any $t \in C$, $(\mathcal{X}^{an}, \mathcal{X}_t^{an})$ is log canonical. We can indeed get a \mathbb{Q} -Fano extension \mathcal{X}^s in (1) such that the divisor \mathcal{X}_t^s as a valuation over $\tilde{\mathcal{X}}^{an} := \mathcal{X} \times_C C'$ has the discrepancy $a(\mathcal{X}_t^s; \tilde{\mathcal{X}}^{an}) = 0$.*

Proof. Let $\phi : C' \rightarrow C$ be a finite base change such that $\mathcal{X} \times_C C'$ yields a semi-stable log resolution $\pi : \mathcal{Y} \rightarrow \mathcal{X} \times_C C'$. We can assume \mathcal{Y} yields an exceptional divisor A where is π -ample. Let π^* be the restriction of π over C^* . By abuse of notation, we will identify $C = C'$ and $\mathcal{X}^* = \mathcal{X} \times_C C^*$.

Write $\pi^*(K_{\mathcal{X}^*}) + F^* = K_{\mathcal{Y}^*} + E^*$, where $E^*, F^* > 0$ are effective divisors without common components, by our assumption the coefficients of E^* are less than 1. Let E, F be the closures of E^* and F^* . Let ϵ be a sufficiently small positive number such that $\mathcal{L}_{\mathcal{Y}} := \pi^*\mathcal{L} + \epsilon A$ is ample. We write $A = A_1 + A_2$, such that A_2 precisely consists of the components which are over 0. By perturbing $\mathcal{L}_{\mathcal{Y}}$ and reordering the components, we can assume that if we write

$$K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + E \sim_{\mathbb{Q}, C} (\epsilon A_1 + F) + (\epsilon A_2 + B)$$

with B supported on the fibers $\mathcal{Y}_0 = \sum E_j$ and $B + \epsilon A_2 = \sum_{j=1}^k a_j E_j$, then $a_j > a_1$ if $j > 1$, where E_j ($1 \leq j \leq k$) are all the components of \mathcal{Y}_0 .

Let G be the sum of the prime divisors which are π -exceptional divisors whose centers are in \mathcal{X}^* . By choosing $\epsilon \ll \delta \ll 1$, we assume that $\delta G + \epsilon A_1 \geq 0$ and its support is equal to G . We run $(K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + E + \delta G)$ -MMP over \mathcal{X} with scaling of $\mathcal{L}_{\mathcal{Y}}$, by adding multiples of fibers, we can also assume $a_1 = 0$ in the above formula. Because

$$(K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + E + \delta G) \sim_{\mathbb{Q}, C} (\delta G + \epsilon A_1 + F) + (\epsilon A_2 + B - a_1 \mathcal{Y}_0) \geq 0,$$

whose support restricting on \mathcal{Y}^* contains all the exceptional divisors for $\mathcal{Y}^* \rightarrow \mathcal{X}^*$, we can apply Proposition 1 and conclude this sequence of MMP terminates with a model \mathcal{X}^m satisfying that $K_{\mathcal{X}^m} + \mathcal{L}^m \sim_{\mathbb{Q}, C} 0$, the only remaining component over 0 is the birational transform of E_1 and \mathcal{L}^m is big and nef. Thus we can define \mathcal{X}^s to be $\text{Proj}R(\mathcal{X}^m/C, -K_{\mathcal{X}^m})$. Over C^* , since $\mathcal{Y}^* \dashrightarrow \mathcal{X}^{m*}$ contracts the same components as the ones of $\mathcal{Y}^* \rightarrow \mathcal{X}^*$, thus $\mathcal{X}^{m*} (:= \mathcal{X}^m \times_C C^*) \dashrightarrow \mathcal{X}^*$ is isomorphic in codimension 1. Hence we see that

$$\mathcal{X}^* = \text{Proj}R(\mathcal{X}^*/C^*, \mathcal{L}|_{\mathcal{X}^*}) \cong \text{Proj}R(\mathcal{X}^{m*}/C^*, \mathcal{L}^m|_{\mathcal{X}^{m*}}) = \mathcal{X}^{s*}.$$

Representing $\mathcal{L}_{\mathcal{Y}}$ by a general \mathbb{Q} -divisor, we can assume $(\mathcal{Y}, E + \delta G + \mathcal{Y}_0 + \mathcal{L}_{\mathcal{Y}})$ is dlt, The MMP sequence is also a sequence of $(K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0 + \mathcal{L}_{\mathcal{Y}})$ -MMP, thus $(\mathcal{X}^m, \mathcal{X}_0^m + \mathcal{L}^m)$ is dlt. This implies that $(\mathcal{X}^m, \mathcal{X}_0^m)$ is dlt since \mathcal{X}^m is \mathbb{Q} -factorial. As \mathcal{X}_0^m is irreducible, this indeed says $(\mathcal{X}^m, \mathcal{X}_0^m)$ is plt and so $(\mathcal{X}^s, \mathcal{X}_0^s)$ is plt. By adjunction, we know \mathcal{X}_0^s is klt. This finishes the proof of (1).

For (2), we apply the same line of argument. We first choose $\mathcal{X} = \mathcal{X}^{an}$. Then we know that we can write $K_{\mathcal{Y}} + E = \pi^*(K_{\mathcal{X}^{an}}) + F + B$, where B is over $\{0\}$. Since $(\mathcal{X}^{an}, \mathcal{X}_0^{an})$ is log canonical, \mathcal{X}^{an} is canonical along \mathcal{X}_0^{an} . So $B \geq 0$, whose support is the union of those divisors $E_j \subset \mathcal{Y}_0$ such that $a(E_j, \mathcal{X}^{an}) > 0$. Now we have

$$K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + E + \delta G \sim_{\mathbb{Q}, \mathcal{X}} B + F + \epsilon A_1 + \epsilon A_2 + \delta G,$$

whose vertical part over $\{0\}$ is $B + \epsilon A_2$. Thus by choosing $0 < \epsilon \ll 1$, we can assume after a small suitable perturbation, the divisor E_1 having the smallest coefficient a_1 is not contained in $\text{Supp}(B)$, i.e., it satisfies that $a(E_1, \mathcal{X}^{an}) = 0$. Then from the proof of (1), \mathcal{X}_0^m will be the birational image of such E_1 . \square

As we mentioned, the component E_1 constructed in this way was first investigated in [Kol07a] and [HX09]. Since any \mathbb{Q} -Fano variety is rationally connected (see [Zha06]), we indeed give a new proof of [HX09, Theorem 3.1] which was originally obtained by applying Hacon-McKernan's extension theorem as in [HM07].

3 Donaldson-Futaki invariant and K-stability

In this section, we will concentrate on the study of Donaldson-Futaki invariants of a test configuration, which is algebraically defined by Donaldson [Don02]. In the first subsection, we give a small modification of the original definition of K -polystability. In the second subsection, we verify that given any test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \rightarrow \mathbb{A}^1$, its Donaldson-Futaki invariant coincides with the Donaldson-Futaki intersection number of the natural compactification $(\bar{\mathcal{X}}^{tc}, \bar{\mathcal{L}}^{tc}) \rightarrow \mathbb{P}^1$. This characterization of the Donaldson-Futaki invariants as intersection numbers is well known to the experts (see, e.g., [Oda09, Wan11]).

3.1 Normal test configuration

In the original definition of K -polystability, it is required that for any test configuration, its Donaldson-Futaki invariant being zero implies the triviality $\mathcal{X}^{tc} \cong X \times \mathbb{A}^1$. However, we believe for the sensible definition, we should only consider the normal test configuration. We illustrate this by calculating the following simple example.

Example 2. Let $(X, L) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. Consider the test configuration $\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{A}^1 = \mathbb{P}(x, y, z, w) \times k[a]$ given by

$$I = (a^2(x+w)w - z^2, ax(x+w) - yz, xz - ayw, y^2w - x^2(x+w)).$$

(cf. [Har77, III.9.8.4]). The \mathbb{G}_m action on it is just sending

$$\mathcal{X} \times \mathbb{G}_m \rightarrow \mathcal{X} : (x, y, z, w; a) \times \{t\} \rightarrow (x, y, tz, w; at).$$

Then for $a = 0$, the special fiber has the ideal

$$I_0 = (z^2, yz, xz, y^2w - x^2(x + w)).$$

Geometrically, \mathcal{X}_t^{tc} is a cubic curve in \mathbb{P}^3 . They degenerate to the special fiber \mathcal{X}_0^{tc} which is a plane nodal cubic curve in $\mathbb{P}^2 = \mathbb{P}(x, y, w)$ with an embedded point at $(0, 0, 0, 1)$.

For $k \gg 0$, we have

$$h^0(\mathbb{P}^1, kL) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3k)) = 3k + 1, \quad \text{and} \quad H^0(\mathcal{X}_0, k\mathcal{L}_0) = V_1 \oplus V_2,$$

where

$$V_1 \cong H^0(\mathcal{X}_0^{\text{red}}, \mathcal{O}_{\mathbb{P}(x, y, w)}(k)|_{\mathcal{X}_0^{\text{red}}})$$

and V_2 is the one dimensional space spanned by $z \cdot w^{k-1}$ (or $z \cdot f(x, y, w)$ for any homogeneous polynomial of degree $k - 1$ such that $f(0, 0, 1) \neq 0$). As the total weight of V_1 is 0 and the total weight of V_2 is 1, we conclude that $b_0 = b_1 = 0$.

Remark 5. 1. As \mathbb{P}^1 admits Kähler-Einstein metric, it should be ‘polystable’ for the right definition of ‘stable’. On the other hand, in [Sto09], J. Stoppa claimed a proof of the K-stability of varieties with Kähler-Einstein metric under the original definition, namely without assuming the normality of the test configuration. However, he made a mistake on the calculation of the Donaldson-Futaki invariant for the ‘degenerate case’.

More precisely, the formula (3.7) of the proof of Proposition 3.3 in [Sto09] is false because multiplying sections $H^0(\mathcal{X}_0^{\text{red}}, \mathcal{L}^{k-1}|_{\mathcal{X}_0^{\text{red}}})$ by a nilpotent element is not always an injection in general. As a check, in the above example, the product of z with any nonzero section of the form

$$g(x, y, w) \in H^0(\mathcal{X}_0^{\text{red}}, \mathcal{O}_{\mathbb{P}^3}(k-1)|_{\mathcal{X}_0^{\text{red}}}),$$

where $g(x, y, w)$ does not have the term w^{k-1} , is zero.

There is also similar overlooking in [PT09]. Corollary 2 there said that the properness of K-energy implies Donaldson-Futaki invariant is positive for any test configuration. The case missing in their proof in section 3.3 is when $\lim_{t \rightarrow 0} \text{Osc}(\phi_t) < \infty$ and the central fibre is generically reduced as the above example shows.

2. In fact, we can easily make such examples for any polarized varieties (X, L) in the following way: Let $(\mathcal{X}^\sharp, \mathcal{L}^\sharp) = (X, L) \times \mathbb{A}^1$. Let \mathcal{L}_X^\sharp be the total space of the line bundle. We choose $x_1, x_2 \in X$ two different points and take an isomorphism $\phi : \mathcal{L}_{x_1}^\sharp \cong \mathcal{L}_{x_2}^\sharp$. We note that the choice of such ϕ is up to a \mathbb{G}_m -action, we just fix one of them. Then it follows from [Art91, 3.1] that there exists an algebraic space \mathcal{X} by identifying two points $x_1, x_2 \in \mathcal{X}^\sharp$ with a line bundle \mathcal{L}^\sharp obtained by gluing the two fibers $\mathcal{L}_{x_1}^\sharp$ and $\mathcal{L}_{x_2}^\sharp$ via ϕ . Thus we have a morphism $n : \mathcal{X}^\sharp \rightarrow \mathcal{X}^{tc}$ with $n^*(\mathcal{L}^{tc}) \cong \mathcal{L}^\sharp$. It is easy to see that \mathcal{L}^{tc} is indeed an ample line bundle and \mathcal{X} is an integral variety. Hence $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ gives a test configuration of (X, L) . A similar calculation as above shows that $\text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = 0$ (also see the equality condition for [ALV11, 3.9]). Therefore, for the K-(poly)-stability notation for a general polarized S_2 (here S_2 means Serre’s condition) variety (X, L) as studied in [Oda08, Oda09], we need to assume the test configuration to be S_2 for the statement that $\text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = 0$ implies the test configuration is trivial. For how this pathological test configuration occurs in the context of one parameter group $\mathbb{G}_m \subset \text{PLG}(N+1)$ acting on $X \subset \mathbb{P}^N$, see the ‘degenerate case’ in [Sto09, 3.3].

3. As far as we can see, in most of the published literature, including [RT07], [Oda08] and [Oda11], the same arguments of proving the results on K -stability for certain classes of varieties work, once we replace the definition there by our new definition. More precisely, for any nontrivial normal test configuration \mathcal{X}^{tc} , there is a semi-test configuration \mathcal{Y}^{tc} with equivariant morphisms $p: \mathcal{Y}^{tc} \rightarrow \mathcal{X}^{tc}$ and $q: \mathcal{Y}^{tc} \rightarrow X \times \mathbb{A}^1$ such that q is not the trivial morphism. Therefore, q gives an exceptional divisor E over $X \times \mathbb{A}^1$. Then their calculations can be carried out by using this exceptional divisor.

3.2 Intersection formula for the Donaldson-Futaki invariant

Given any test configuration $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$, we first compactify it by gluing $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ with $(X \times (\mathbb{P}^1 \setminus \{0\}), p_1^*L)$. Then we will show that the Donaldson-Futaki invariant is equal to the DF intersection number on this compactified space. The same formula appeared before (see e.g. [Oda09], [Wan11]). We include a proof here using Donaldson's argument.

Example 3. \mathbb{G}_m acts on $(X, L^{-1}) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$ by

$$t \circ ([Z_0, Z_1], \lambda(Z_0, Z_1)) = ([Z_0, tZ_1], \lambda(Z_0, tZ_1)).$$

In particular, the \mathbb{G}_m -weights on

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_0, \mathcal{O}_{\mathbb{P}^1}(1)|_0, \mathcal{O}_{\mathbb{P}^1}(-1)|_\infty \text{ and } \mathcal{O}_{\mathbb{P}^1}(1)|_\infty$$

are 0, 0, 1 and -1. Let $\tau_0 = Z_1$, $\tau_\infty = Z_0$ be the holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Then the \mathbb{G}_m -weights of τ_0 and τ_∞ are -1 and 0.

Take $\bar{\mathcal{X}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ and $\bar{\mathcal{L}} = \mathcal{O}_{\bar{\mathcal{X}}}(1) = \mathcal{O}_{D_\infty}$, where D_∞ is the divisor at infinity. We see that $(\mathcal{X}^{tc} := \bar{\mathcal{X}} \setminus \mathbb{P}_\infty^1, \mathcal{L}^{tc} := \bar{\mathcal{L}}|_{\mathcal{X}^{tc}})$ yields a test configuration of (X, L) . Then $H^0(\mathbb{P}^1, L^{\otimes k})$ is of dimension $d_k = k + 1$ and by the calculation in the first paragraph the total \mathbb{G}_m -weight of $H^0(\mathbb{P}^1, L^{\otimes k})$ is $w_k = -\frac{1}{2}(k^2 + k)$. We know $D_\infty^2 = -1$ and $K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty = 1$. So

$$w_k = \frac{D_\infty^2}{2}k^2 + \left(\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1 \right) k \quad \text{and} \quad \text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = \frac{D_\infty^2}{2} - \left(\frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1 \right) (= 0).$$

This example can be generalized to more general cases (see (5), (6)) by using Donaldson's argument in the following way (also see the proof of Proposition 4.2.1 in [Don02]).

First note that, after identifying the fiber \mathcal{X}_1^{tc} over $\{1\}$ and X , we have an equivariant isomorphism:

$$(\mathcal{X}^{tc} \setminus \mathcal{X}_0^{tc}, \mathcal{L}^{tc}) \cong (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^*L)$$

by $(p, a, s) \rightarrow (a^{-1} \circ p, a, a^{-1} \circ s)$. Therefore, \mathbb{G}_m acts on the right hand side by

$$t \circ (\{p\} \times \{a\}, s) = (\{p\} \times \{ta\}, s)$$

for any $p \in X$, $a \in \mathbb{A}^1$ and $s \in \mathcal{L}_p^{tc}$. The gluing map is given by

$$\begin{aligned} & \begin{array}{ccc} (\mathcal{X}^{tc}, \mathcal{L}^{tc}) & & (X \times \mathbb{P}^1 \setminus \{0\}, p_1^*L) \\ \cup & & \cup \\ (\mathcal{X}^{tc} \setminus \mathcal{X}_0^{tc}, \mathcal{L}^{tc}) & \longrightarrow & (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^*L) \end{array} \\ & (p, a, s) \longmapsto (\{a^{-1} \circ p\} \times \{a\}, a^{-1} \circ s), \end{aligned}$$

where \mathbb{G}_m only acts by multiplication on the factor $\mathbb{P}^1 \setminus \{0\}$ of $(X \times \mathbb{P}^1 \setminus \{0\}, p_1^*L)$.

Using the above gluing map, we get a compact complex manifold projective over \mathbb{P}^1 : $\bar{\pi}: (\bar{\mathcal{X}}^{tc}, \bar{\mathcal{L}}^{tc}) \rightarrow \mathbb{P}^1$. In the following of this subsection, we will denote $(\bar{\mathcal{X}}^{tc}, \bar{\mathcal{L}}^{tc})$ by $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ for simplicity. Note that there exists an integer N , such that $\bar{\mathcal{M}} = \bar{\mathcal{L}} \otimes \bar{\pi}^*(\mathcal{O}_{\mathbb{P}^1}(N \cdot \{\infty\}))$ is ample on $\bar{\mathcal{X}}$ (cf. [KM98, 1.45]).

We need the following weak form of the Riemann-Roch formula whose proof is well known.

Lemma 2. *Let X be an n -dimensional normal projective variety and L an ample divisor on X then*

$$\dim H^0(X, L^{\otimes k}) = \frac{L^n}{n!} k^n + \frac{1}{2} \frac{K_X^{-1} \cdot L^{n-1}}{(n-1)!} k^{n-1} + O(k^{n-2}).$$

We define

$$d_k = \dim H^0(X, L^{\otimes k}) =: a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

Proposition 3. *Assume \mathcal{X}^{tc} is normal, then*

$$\mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = \mathrm{DF}(\bar{\mathcal{X}}/\mathbb{P}^1, \bar{\mathcal{L}}) \quad (4)$$

Proof. For $k \gg 0$, by Serre Vanishing Theorem, we have two exact sequences:

$$\begin{array}{ccccccc} & A & & B & & C & \\ & \parallel & & \parallel & & \parallel & \\ 0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\mathcal{X}_0^{tc})) & \xrightarrow{\otimes \sigma_0} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\bar{\mathcal{X}}_0, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0^{tc}}) & \longrightarrow 0 \\ & \parallel & & \parallel & & \parallel & \\ 0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\bar{\mathcal{X}}_\infty)) & \xrightarrow{\otimes \sigma_\infty} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\bar{\mathcal{X}}_\infty, \bar{\mathcal{M}}^{\otimes k}|_{\bar{\mathcal{X}}_\infty}) & \longrightarrow 0, \\ & A & & B & & D & \end{array}$$

where σ_0, σ_∞ are sections of $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ which are pull back of the divisors $\{0\}, \{\infty\}$ on \mathbb{P}^1 .

We can assume the \mathbb{G}_m -weights of σ_0 and σ_∞ are -1 and 0. Note the first terms in the two exact sequences are the same as $A := H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-1))$. We have the equation: $w_B = w_A - d_A + w_C = w_A + w_D$, where we denote d_A and w_A to be the dimension and the \mathbb{G}_m -weight of the vector space A and similarly for d_B, w_C etc. Since the \mathbb{G}_m -weight of $\mathcal{O}_{\mathbb{P}^1}(1)|_\infty$ is -1 and \mathbb{G}_m acts on $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ trivially, we have $w_D = -kN \dim H^0(\bar{\mathcal{X}}_\infty, \bar{\mathcal{L}}^{\otimes k}|_{\bar{\mathcal{X}}_\infty})$. So we get

$$w_C = d_A + w_D = d_B - d_C - kNd_D = d_B - (kN + 1)d_C,$$

Since \mathbb{G}_m acts trivially on $\mathcal{O}_{\mathbb{P}^1}(1)|_0$, we get the \mathbb{G}_m -weight on $H^0(\mathcal{X}_0^{tc}, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0^{tc}}) = H^0(\mathcal{X}_0^{tc}, \mathcal{L}^{tc \otimes k}|_{\mathcal{X}_0^{tc}})$:

$$w_k = \dim H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) - (kN + 1) \dim H^0(\mathcal{X}_0^{tc}, \mathcal{L}^{tc \otimes k}|_{\mathcal{X}_0^{tc}}).$$

Expanding w_k , we get:

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$$

with

$$b_0 = \frac{\bar{\mathcal{M}}^{n+1}}{(n+1)!} - Na_0 = \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)!}, \text{ and} \quad (5)$$

$$b_1 = \frac{1}{2} \frac{K_{\bar{\mathcal{X}}}^{-1} \cdot \bar{\mathcal{M}}^n}{n!} - Na_1 - a_0 = \frac{1}{2} \frac{K_{\bar{\mathcal{X}}}^{-1} \cdot \bar{\mathcal{L}}^n}{n!} - a_0. \quad (6)$$

By substituting the coefficients into (2), we get

$$\frac{a_1 b_0 - a_0 b_1}{a_0^2} = \frac{1}{(n+1)! a_0} \left(\frac{a_1}{a_0} \bar{\mathcal{L}}^{n+1} - \frac{n+1}{2} K_{\bar{\mathcal{X}}}^{-1} \cdot \bar{\mathcal{L}}^n \right) + 1, \quad (7)$$

which is the same as (4). \square

Remark 6. 1. As the above proof shows, Donaldson's formula of Futaki invariant in the toric case (Proposition 4.2.1 in [Don02]) is a special example of the intersection formula.

2. This intersection formula is related to the interpretation of Donaldson-Futaki invariant as the CM-weight in [PT06]. It was extensively used in [Oda08], [RT07], etc.
3. When $(\mathcal{X}^{tc}, \mathcal{L}^{tc}) \rightarrow \mathbb{A}^1$ is a test configuration, where we only assume \mathcal{L}^{tc} to be relative big and semi-ample \mathbb{Q} -line bundle, this definition of Donaldson-Futaki invariant using intersection numbers $\text{DF}(\bar{\mathcal{X}}^{tc}/\mathbb{P}^1, \bar{\mathcal{L}}^{tc})$ still coincides with the definition via computing the \mathbb{G}_m -weights of cohomological groups as in [ALV11]. For more details, see [RT07] and [ALV11].

Using this intersection formula, in the following work we need the higher dimensional analogue of the Zariski's lemma for surfaces.

Lemma 3. *Let $\mathcal{X} \rightarrow C$ be a projective dominant morphism from a n -dimensional normal variety to a proper smooth curve. Let E be a \mathbb{Q} -divisor which supports on some fiber \mathcal{X}_0 . Let $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$ be $n-2$ nef divisors on \mathcal{X} . Then*

$$E^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-2} \leq 0.$$

If all \mathcal{L}_i 's are ample, then the equality holds if and only if $E = t\mathcal{X}_0$ for some $t \in \mathbb{Q}$.

Proof. When $n = 2$, this is the well-known Zariski lemma (see e.g. [BHPV95, III.8.2]). We note that since E is not necessarily \mathbb{Q} -Cartier, here we have to use the intersection theory on normal surfaces developed by Mumford (see [Mum61]). For $n > 2$, replacing \mathcal{L} by a multiple, we assume \mathcal{L} to be very ample. Adding a suitable multiple of the fiber, we can assume $E \geq 0$ and its support does not contain $\text{Supp}(\mathcal{X}_0)$. Therefore, we can cut \mathcal{X} by $n-2$ general sections in $|\mathcal{L}|$ and reduce the question to the case when $n = 2$. \square

4 Decreasing of DF intersection number for the log canonical modification

Let $(\mathcal{X}, \mathcal{L}) \rightarrow C$ be a polarized generic \mathbb{Q} -Fano family. It is easy to see all our operations will be local over C , so to simplify the notation, without loss of generality we will just denote one degenerate fiber to be \mathcal{X}_0 and argue in a neighborhood of it.

In this section, we aim to verify Theorem 3. In fact, we will calculate the DF intersection numbers on the log canonical modification $\mathcal{X}^{lc} \rightarrow \mathcal{X}$. We start from the line bundle $\pi^{lc*}\mathcal{L}$ which is the pull back of the polarization on \mathcal{X} , whose DF intersection number is equal to the original one. Since $K_{\mathcal{X}^{lc}}$ is relative ample, if we deform $\pi^{lc*}\mathcal{L}$ along the direction $K_{\mathcal{X}^{lc}}$ sufficiently small amount then we get an ample bundle on \mathcal{X}^{lc} . As this is a deformation along the canonical class, we can show the DF intersection numbers decrease along this deformation.

Assume \mathcal{X} has a semi-stable resolution \mathcal{Y} . Let $\pi^{lc} : \mathcal{X}^{lc} \rightarrow \mathcal{X}$ be the log canonical modification (cf. Proposition 2) such that $\mathcal{X}^{lc} \times_C C^*$ is isomorphic to $\mathcal{X} \times_C C^*$.

Proposition 4. *With the above notations. If \mathcal{X}^{lc} is not isomorphic to \mathcal{X} , then we can choose a polarization \mathcal{L}^{lc} on \mathcal{X}^{lc} such that*

$$\text{DF}(\mathcal{X}^{lc}/C, \mathcal{L}^{lc}) < \text{DF}(\mathcal{X}/C, \mathcal{L}).$$

Proof. By definition, $K_{\mathcal{X}^{lc}}$ is π^{lc} -ample. We choose the relatively π^{lc} -ample \mathbb{Q} -divisor

$$E = K_{\mathcal{X}^{lc}} + \frac{1}{r}\pi^{lc*}(\mathcal{L}).$$

Then E is \mathbb{Q} -linearly equivalent to a divisor which only supports on \mathcal{X}_0^{lc} . Since for sufficiently small rational ϵ , $\mathcal{L}_t^{lc} = \pi^{lc*}\mathcal{L} + tE$ is ample for any $0 < t < \epsilon$, we see that $(\mathcal{X}^{lc}, \mathcal{L}^{lc}) \rightarrow C$ is also a polarized generic \mathbb{Q} -Fano family.

Using the formula, we compute the derivative

$$\begin{aligned} \frac{d}{dt} \text{DF}(\mathcal{X}^{lc}/C, \mathcal{L}_t^{lc}) &= C_0 \cdot \left(\frac{n(n+1)}{r} (\mathcal{L}_t^{lc})^n \cdot E + n(n+1) K_{\mathcal{X}^{lc}} \cdot (\mathcal{L}_t^{lc})^{n-1} \cdot E \right) \\ &= C_1 \cdot (\mathcal{L}_t^{lc})^{n-1} \cdot E \cdot \left(\frac{1}{r} \mathcal{L}_t^{lc} + K_{\mathcal{X}^{lc}} \right) = C_2 \cdot (\mathcal{L}_t^{lc})^{n-1} \cdot E^2, \end{aligned}$$

where C_0, C_1 and C_2 are positive numbers. By Lemma 3, the intersection $(\mathcal{L}_t^{lc})^{n-1} \cdot E^2 \leq 0$ and it is zero if and only if $E = K_{\mathcal{X}^{lc}} + \frac{1}{r} \pi^{lc*} \mathcal{L}$ is \mathbb{Q} -linearly equivalent to $t\mathcal{X}_0^{lc}$ for some t . But this implies that $\mathcal{X}^{lc} \cong \mathcal{X}$ since $K_{\mathcal{X}^{lc}} \sim_{\mathbb{Q}, \mathcal{X}} E$ is π^{lc} -ample. \square

Proof of Theorem 3. First we can take the base change $\mathcal{X} \times_C C'$ such that its normalization $\tilde{\mathcal{X}}$ admits a semi-stable reduction \mathcal{Y} . In particular $\tilde{\mathcal{X}}_0$ is reduced. Let $\phi_{\mathcal{X}} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be the natural finite morphism and $\tilde{\mathcal{L}} = \phi_{\mathcal{X}}^* \mathcal{L}$. We first note that

Claim 1.

$$\deg(C'/C) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) \geq \text{DF}(\tilde{\mathcal{X}}/C', \tilde{\mathcal{L}}).$$

Furthermore, the equality holds if and only if \mathcal{X}_0 is reduced.

Indeed, by the pull-back formula for the log differential, we have $K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0 = f^*(K_{\mathcal{X}} + \text{red}(\mathcal{X}_0))$ and $K_{C'} + \{0'\} = \phi^*(K_C + \{0\})$. So

$$K_{\tilde{\mathcal{X}}/C'} = f^*(K_{\mathcal{X}/C} + (\text{red}(\mathcal{X}_0) - \mathcal{X}_0))$$

and the claim follows from the projection formula.

Now it follows from Proposition 2 that the log canonical modification $\pi^{lc} : \mathcal{X}^{lc} \rightarrow \tilde{\mathcal{X}}$ exists. And then Proposition 4 shows that

$$\text{DF}(\tilde{\mathcal{X}}/C', \tilde{\mathcal{L}}) \geq \text{DF}(\mathcal{X}^{lc}/C', \mathcal{L}^{lc}).$$

If $(\mathcal{X}, \mathcal{X}_0)$ is log canonical, then \mathcal{X}_0 is reduced and $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$ is log canonical (cf. [KM98, 5.20]), which implies $\mathcal{X}^{lc} \cong \tilde{\mathcal{X}}$, therefore the equality holds.

Conversely, $\deg(\phi) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) = \text{DF}(\mathcal{X}^{lc}/C', \mathcal{L}^{lc})$ is equivalent to saying the above two inequalities are indeed equalities. By Proposition 4 and the above claim, this holds only if \mathcal{X}_0 is reduced and $\mathcal{X}^{lc} \cong \tilde{\mathcal{X}}$ which implies $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$ is log canonical. Since

$$\phi_{\mathcal{X}}^*(K_{\mathcal{X}} + \mathcal{X}_0) = K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0,$$

it follows that $(\mathcal{X}, \mathcal{X}_0)$ is also log canonical (see [KM98, 5.20]). \square

5 MMP with scaling

In this section, we aim to prove Theorem 4. We will apply the idea that the Donaldson-Futaki intersection numbers decrease if we deform the polarization along the direction of the canonical class of the total family in ‘a long time’ process. To keep the deformed line bundle being a polarization, we have to do a sequence of surgeries on the family. In algebraic geometry, this surgery is given by the MMP with scaling (see [BCHM10] and Subsection 2.3).

5.1 Running MMP

After replacing \mathcal{L}^{lc} by $m\mathcal{L}^{lc}$ for sufficiently large m , we can assume $\mathcal{H}^{lc} = \mathcal{L}^{lc} - K_{\mathcal{X}^{lc}}$ is relatively ample. Let $\lambda_0 = 1$. We let $\mathcal{X}^0 = \mathcal{X}^{lc}$, $\mathcal{L}^0 = \mathcal{L}^{lc}$ and $\mathcal{H}^0 = \mathcal{H}^{lc}$. Then $K_{\mathcal{X}^0} + \lambda_0\mathcal{H}^0 = \mathcal{L}^0$ is relatively ample.

Given an exceptional divisor E , if its center dominates C then $a(E, \mathcal{X}^0) > -1$ because \mathcal{X}^* is klt; if its center is vertical over C , then $a(E, \mathcal{X}^0) \geq 0$, since $(\mathcal{X}^0, \mathcal{X}_t^0)$ is log canonical for any t in C . In particular, \mathcal{X}^0 is klt. To simplify the family, we run a sequence of $K_{\mathcal{X}^0}$ -MMP over C with scaling of \mathcal{H}^0 as in Subsection 2.3. So we obtain a sequence of models

$$\mathcal{X}^0 \dashrightarrow \mathcal{X}^1 \dashrightarrow \dots \dashrightarrow \mathcal{X}^k.$$

Recall that, as in Subsection 2.3, we have a sequence of critical value of scaling factors

$$\lambda_{i+1} = \min\{\lambda \mid K_{\mathcal{X}^i} + \lambda\mathcal{H}^i \text{ is nef over } C\}$$

with $1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_k > \lambda_{k+1} = \frac{1}{r+1}$. Note that $\frac{1}{r+1}$ is the pseudo-effective threshold of $K_{\mathcal{X}^0}$ with respect to \mathcal{H}^0 over C , since it is the pseudo-effective threshold for the generic fiber. Any \mathcal{X}^i appearing in this sequence of $K_{\mathcal{X}^0}$ -MMP with scaling of \mathcal{H}^0 is a relative weak log canonical model of $(\mathcal{X}^0, t\mathcal{H}^0)$ for any $t \in [\lambda_i, \lambda_{i+1}]$ (see [BCHM10, 3.6.7] for the definition of weak log canonical model).

For $\lambda > \frac{1}{r+1}$, we denote by

$$\mathcal{L}_\lambda = \frac{r}{\lambda(r+1) - 1} (K_{\mathcal{X}^0} + \lambda\mathcal{H}^0). \quad (8)$$

Let \mathcal{L}_λ^i (resp. \mathcal{H}^i) be the push forward of \mathcal{L}^0 (resp. \mathcal{H}^0) to \mathcal{X}^i . As is clear from the context, this should not be confused with the i -th power or intersection product of \mathcal{L}_λ (resp. \mathcal{H}). We note that by definition $\mathcal{L}_1^0 = \mathcal{L}^0$.

Given a λ , we will be interested in those i which satisfy that $\lambda_i \geq \lambda \geq \lambda_{i+1}$. Note that

$$K_{\mathcal{X}^i} + \frac{1}{r}\mathcal{L}_\lambda^i = \frac{\lambda(r+1)}{\lambda(r+1) - 1} \left(K_{\mathcal{X}^i} + \frac{1}{r+1}\mathcal{H}^i \right) \quad (9)$$

Lemma 4. $-rK_{\mathcal{X}^k} \sim_{\mathbb{Q}, C} \mathcal{L}_{\lambda_k}^k$ is big and semi-ample over C .

Proof. Since $\lambda_k > \lambda_{k+1} = \frac{1}{r+1}$, by (9),

$$K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}_{\lambda_k}^k \sim_{\mathbb{Q}} \frac{\lambda_k(r+1)}{\lambda_k(r+1) - 1} \left(K_{\mathcal{X}^k} + \frac{1}{r+1}\mathcal{H}^k \right).$$

This line bundle is relatively nef over C and its restriction to the generic fiber is trivial, so it is \mathbb{Q} -linearly equivalent to a linear sum of components of \mathcal{X}_0^k . By its nefness, we can apply Lemma 3 to get

$$K_{\mathcal{X}^k} + \frac{1}{r}\mathcal{L}_{\lambda_k}^k \sim_{\mathbb{Q}, C} 0.$$

By (8), $\mathcal{L}_{\lambda_k}^k$ is proportional to $K_{\mathcal{X}^k} + \lambda_k\mathcal{H}^k$ which is big because $\lambda_k > \frac{1}{r+1}$. From the relative base-point free theorem (cf. Theorem 3.3 in [KM98]), it is semi-ample over C . \square

By the above Lemma, we can define

$$\mathcal{X}^{an} = \text{Proj} R(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \text{Proj} R(\mathcal{X}^k/C, -rK_{\mathcal{X}^k/C}).$$

Since $(\mathcal{X}^0, \mathcal{X}_0^0)$ is log canonical and $\mathcal{X}_0^0 = (f \circ \pi^{lc})^*(\{0\})$, this is also a sequence of $(K_{\mathcal{X}^0} + \mathcal{X}_0^0)$ -MMP and thus $(\mathcal{X}^k, \mathcal{X}_0^k)$ is log canonical which implies that $(\mathcal{X}^{an}, \mathcal{X}_0^{an})$ is log canonical as well.

5.2 Decreasing of DF-intersection number

For any $\lambda > \frac{1}{r+1}$, the restriction of $K_{\mathcal{X}^0} + \lambda \mathcal{H}^0$ over C^* is ample. So the MMP with scaling does not change $\mathcal{X}^0 \times_C C^*$, i.e., $\mathcal{X}^0 \times_C C^* \cong \mathcal{X}^i \times_C C^*$ for any $i \leq k$.

Note that by the above lemma and projection formula,

$$\mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda^k}^k) = \mathrm{DF}(\mathcal{X}^k/C, -rK_{\mathcal{X}^k}) = \mathrm{DF}(\mathcal{X}^{an}/C, -rK_{\mathcal{X}^{an}}).$$

So Theorem 4 follows from the following Proposition.

Proposition 5. *With the notations as above, we have*

$$\mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}^0) \geq \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda^k}^k) = \mathrm{DF}(\mathcal{X}^k/C, -rK_{\mathcal{X}^k}).$$

The first equality holds if and only if $h : \mathcal{X}^0 \dashrightarrow \mathcal{X}^k$ is an isomorphism.

5.2.1 Decreasing of DF on a fixed model

Assume $\mathcal{X}_0^0 = \sum_{i=1} E_i$, where E_i 's are the prime divisors. Since $(\mathcal{X}^0, \mathcal{L}_\lambda^0) \times_C C^*$ is isomorphic to $(\mathcal{X}^0 \times_C C^*, -rK_{\mathcal{X}^0 \times_C C^*})$, there exist $a_j(\lambda) \in \mathbb{R}$ such that

$$K_{\mathcal{X}^0} + \frac{1}{r} \mathcal{L}_\lambda^0 \sim_{\mathbb{R}, C} \sum_{j \in I} a_j(\lambda) E_j.$$

On \mathcal{X}^i , for any rational number $\lambda > \frac{1}{r+1}$ satisfying $\lambda_i \geq \lambda \geq \lambda_{i+1}$, we know \mathcal{L}_λ^i is a big and semi-ample. Let \mathcal{Z}_λ be the relative log canonical model of $(\mathcal{X}^0, \lambda \mathcal{H}^0)$ over C . Then there is a morphism $\pi_\lambda : \mathcal{X}^i \rightarrow \mathcal{Z}_\lambda$ and an relatively ample \mathbb{Q} -divisor \mathcal{M}_λ on \mathcal{Z}_λ whose pull back is

$$\frac{r}{\lambda(r+1)-1} (K_{\mathcal{X}^i} + \lambda \mathcal{H}^i) = \mathcal{L}_\lambda^i.$$

Lemma 5. *If $\lambda_i \geq a > b \geq \lambda_{i+1}$ and $b > \frac{1}{r+1}$, then $\mathrm{DF}(\mathcal{X}^i, \mathcal{L}_a^i) \geq \mathrm{DF}(\mathcal{X}^i, \mathcal{L}_b^i)$. The inequality is strict if there is a rational number $\lambda \in [a, b]$, such that the push forward of $\sum_{j \in I} a_j(\lambda) E_j$ to \mathcal{Z}_λ is not a multiple of the pull back of $0 \in C$ on \mathcal{Z}_λ .*

Proof. We compute the derivative of the Donaldson-Futaki invariants in a similar way as Propostion 4.

$$\begin{aligned} \frac{d}{d\lambda} \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_\lambda^i) &= C_0 \left((\mathcal{L}_\lambda^i)^{n-1} \cdot (\mathcal{L}_\lambda^i)' \cdot \left(\frac{1}{r} \mathcal{L}_\lambda^i + K_{\mathcal{X}^i} \right) \right) \\ &= -\frac{rC_0}{\lambda((r+1)\lambda-1)} (\mathcal{L}_\lambda^i)^{n-1} \cdot \left(K_{\mathcal{X}^i} + \frac{1}{r} \mathcal{L}_\lambda^i \right)^2 \\ &= -\frac{rC_0}{\lambda(\lambda(r+1)-1)} (\mathcal{L}_\lambda^i)^{n-1} \cdot \left(\sum_{j \in I} a_j(\lambda) E_j \right)^2, \end{aligned}$$

where C_0 is a positive constant. Then the lemma follows from Lemma 3. \square

5.2.2 Invariance of DF at contraction or flip points

If $\lambda_{i+1} > \frac{1}{r+1}$, then by the definition of MMP with scaling, we pick up a $K_{\mathcal{X}^i}$ -negative extremal ray $[R]$ in $\mathrm{NE}(\mathcal{X}^i/C)$ such that $R \cdot (K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i) = 0$. we perform a birational transformation:

$$\begin{array}{ccc} \mathcal{X}^i & \xrightarrow{f^i} & \mathcal{Y}^i \\ & \searrow & \swarrow \\ & \mathbb{P}^1, & \end{array}$$

which contracts all curves R' whose classes $[R']$ are in the ray $\mathbb{R}_{>0}[R]$. There are two cases:

1. (Divisorial Contraction) If f^i is a divisorial contraction. Then $\mathcal{X}^{i+1} = \mathcal{Y}^i$. Since f^i is a $(K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i)$ -trivial morphism by the definition of the MMP with scaling, we have

$$K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i = (f^i)^*(K_{\mathcal{Y}^i} + \lambda_{i+1}\mathcal{H}^{i+1}),$$

which implies

$$\mathcal{L}_{\lambda_{i+1}}^i = (f^i)^*\mathcal{L}_{\lambda_{i+1}}^{i+1}.$$

Then it follows from Definition 2 and projection formula that

$$\mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_{i+1}}^i) = \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+1}}^{i+1}).$$

2. (Flipping Contraction) If f^i is a flipping contraction, let $\phi^i : \mathcal{X}^i \dashrightarrow \mathcal{X}^{i+1}$ be the flip.

$$\begin{array}{ccc} \mathcal{X}^i & \xrightarrow{\phi^i} & \mathcal{X}^{i+1} \\ & \searrow f^i & \nearrow f^{i+} \\ & \mathcal{Y}^i & \end{array}$$

$-K_{\mathcal{X}^i}$ is f^i -ample $K_{\mathcal{X}^{i+1}}$ is f^{i+} -ample

As f^i is a $K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i$ -trivial morphism, $K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i = (f^i)^*D_{\mathcal{Y}^i}$ for some divisor $D_{\mathcal{Y}^i}$. Since f^i, f^{i+}, ϕ^i are isomorphisms in codimension one, we also have $K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{H}^{i+1} = (f^{i+})^*D_{\mathcal{Y}^i}$. Therefore, using the intersection formula, we see that

$$\mathrm{DF}(\mathcal{X}^i/C, K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i) = \mathrm{DF}(\mathcal{Y}^i/C, D_{\mathcal{Y}^i}) = \mathrm{DF}(\mathcal{X}^{i+1}/C, K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{H}^{i+1}).$$

Now we can finish the proof of Proposition 5:

Proof.

$$\begin{aligned} & \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0}^0) \geq \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_1}^0) \\ = & \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}_{\lambda_1}^1) \geq \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}_{\lambda_2}^1) \\ & \dots \quad \dots \quad \dots \\ = & \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_i}^i) \geq \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_{i+1}}^i) \\ = & \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+1}}^{i+1}) \geq \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+2}}^{i+1}) \\ & \dots \quad \dots \quad \dots \\ = & \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \mathrm{DF}(\mathcal{X}^k/C, -rK_{\mathcal{X}^k}). \end{aligned}$$

Now we characterize the equality case. Since $-K_{\mathcal{X}^k} \sim_{\mathbb{Q}, C} \frac{1}{r}\mathcal{L}_{\lambda_k}^k$ is relatively nef over C , we conclude that $f^{k-1} : \mathcal{X}^{k-1} \rightarrow \mathcal{X}^k$ can only be a divisorial contraction. Therefore, $h : \mathcal{X}^0 \dashrightarrow \mathcal{X}^k$ contracts at least one divisor if it is not an isomorphism.

Since \mathcal{X}^k is a minimal model of $(\mathcal{X}^0, \frac{1}{r+1}\mathcal{H}^0)$, we know that

$$0 < E = K_{\mathcal{X}^0} + \frac{1}{r+1}\mathcal{H}^0 - h^*(K_{\mathcal{X}^k} + \frac{1}{r+1}\mathcal{H}^k) \sim_{\mathbb{Q}, C} K_{\mathcal{X}^0} + \frac{1}{r+1}\mathcal{H}^0,$$

which is supported on the fiber over 0. It follows from the fact that the support of E is a proper subset of \mathcal{X}_0^0 , $K_{\mathcal{X}^0} + \frac{1}{r+1}\mathcal{H}^0$ is not \mathbb{Q} -linearly equivalent to 0 over C , i.e., the equality condition of Lemma 5 can not hold on \mathcal{X}^{lc} . Thus for a sufficiently small rational number ϵ ,

$$\mathrm{DF}(\mathcal{X}^{lc}/C, \mathcal{L}^{lc}) = \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0}^0) > \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0-\epsilon}^0) \geq \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k).$$

□

6 Revisiting of \mathbb{Q} -Fano extension

From the discussion of the last section, we achieve a model \mathcal{X}^{an} over C with polarization \mathcal{L}^{an} which compactifies \mathcal{X}^*/C^* such that $\mathcal{L}^{an} \sim_{\mathbb{Q}, C} -rK_{\mathcal{X}^{an}}$ and $(\mathcal{X}^{an}, \mathcal{L}_t^{an})$ is log canonical for any $t \in C$. We can not run an MMP directly from \mathcal{X}^{an} to get \mathcal{X}^s . Instead we will resolve \mathcal{X}^{an} again and run MMP. More precisely, by Theorem 7(2), we know that there exists $\phi : C' \rightarrow C$ with a \mathbb{Q} -Fano family \mathcal{X}^s/C' . We will show this is our final \mathbb{Q} -Fano family by verifying the decreasing of the DF intersection number.

Using the notation in Theorem 7, $a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{an}) = 0$ implies

$$a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{an}, \tilde{\mathcal{X}}_0^{an}) = -1,$$

since $\tilde{\mathcal{X}}_0^{an}$ is Cartier and $(\tilde{\mathcal{X}}^{an}, \tilde{\mathcal{X}}_0^{an})$ is log canonical. Then for any number $\lambda \in [0, 1]$, we know that

$$a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{an}, \lambda \tilde{\mathcal{X}}_0^{an}) = -\lambda.$$

In particular, there exists a model $\pi' : \mathcal{X}' \rightarrow \tilde{\mathcal{X}}^{an}$ which precisely extracts the divisor \mathcal{X}_0^s (cf. [BCHM10, 1.4.3]). Since $a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{an}) = 0$, we know $\pi'^*(K_{\tilde{\mathcal{X}}^{an}}) = K_{\mathcal{X}'}$. Then by the projection formula

$$\text{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) = \text{DF}(\tilde{\mathcal{X}}^{an}/C', -K_{\tilde{\mathcal{X}}^{an}}) = \deg(\phi) \cdot \text{DF}(\mathcal{X}^{an}/C, -K_{\mathcal{X}^{an}}).$$

Proposition 6. *We have the inequality*

$$\text{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) \geq \text{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}),$$

and the equality holds if and only if the rational map $\tilde{\mathcal{X}}^{an} \dashrightarrow \mathcal{X}^s$ is an isomorphism.

Proof. By abuse of notation, we identify C and C' , \mathcal{X}^{an} and $\tilde{\mathcal{X}}^{an}$.

Using the intersection formula, we have that

$$\text{DF}(\mathcal{X}'/C, -K_{\mathcal{X}'/C}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}'})^n} (-K_{\mathcal{X}'/C})^{n+1}.$$

Similarly,

$$\text{DF}(\mathcal{X}^s/C, -K_{\mathcal{X}^s/C}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}^s})^n} (-K_{\mathcal{X}^s/C})^{n+1}.$$

Let $p : \hat{\mathcal{X}} \rightarrow \mathcal{X}'$ and $q : \hat{\mathcal{X}} \rightarrow \mathcal{X}^s$ be a common log resolution, and we write

$$(\pi' \circ p)^*(K_{\mathcal{X}^{an}}) = p^*K_{\mathcal{X}'} = q^*K_{\mathcal{X}^s} + E.$$

Since $\mathcal{X}' \dashrightarrow \mathcal{X}^s$ is a birational contraction, by negativity lemma (cf. [KM98, 3.39]) we conclude that $E \geq 0$. For $0 \leq \lambda \leq 1$, let

$$f(\lambda) = (-p^*K_{\mathcal{X}'/C} + \lambda E)^{n+1}.$$

Then for any $0 \leq \lambda \leq 1$

$$\begin{aligned} \frac{df(\lambda)}{d\lambda} &= (n+1)E \cdot (-p^*K_{\mathcal{X}'} + \lambda E)^n \\ &= (n+1)E \cdot (-(1-\lambda)p^*K_{\mathcal{X}'} - \lambda q^*K_{\mathcal{X}^s})^n \\ &\geq 0, \end{aligned}$$

since $-(1-\lambda)p^*K_{\mathcal{X}'} - \lambda q^*K_{\mathcal{X}^s}$ is relatively nef over C . Thus

$$\text{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) \geq \text{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}).$$

We analyze when the equality holds. If $E = 0$, then

$$\mathcal{X}^{an} \cong \text{Proj} R(\mathcal{X}'/C, -K_{\mathcal{X}'/C}) \cong \text{Proj} R(\mathcal{X}^s/C, -K_{\mathcal{X}^s/C}) = \mathcal{X}^s.$$

So we may assume that the effective \mathbb{Q} -divisor E is not equal to 0.

Next we assume \mathcal{X}^{an} is isomorphic to \mathcal{X}^s in codimension 1. Thus for any divisor D on \mathcal{X}^{an} ,

$$R(\mathcal{X}^{an}/C, D) \cong R(\mathcal{X}^s/C, D_{\mathcal{X}^s}),$$

where $D_{\mathcal{X}^s}$ is the push forward of D to \mathcal{X}^s . In particular, if we let $D = -K_{\mathcal{X}^{an}}$, we again have

$$\mathcal{X}^{an} \cong \text{Proj} R(\mathcal{X}^{an}/C, -K_{\mathcal{X}^{an}/C}) \cong \text{Proj} R(\mathcal{X}^s/C, -K_{\mathcal{X}^s/C}) = \mathcal{X}^s.$$

So we can assume that $E > 0$ and \mathcal{X}^{an} is not isomorphic to \mathcal{X}^s in codimension 1, then we claim that:

$$f(0) < f(\lambda) \quad (10)$$

for any $1 > \lambda > 0$.

In fact, since \mathcal{X}_0^s is irreducible, from the above discussion we may assume that there exists a component $E_1 \subset \mathcal{X}_0^{an}$ such that the birational transform \hat{E}_1 of E_1 on $\hat{\mathcal{X}}$ is contracted under $\hat{\mathcal{X}} \rightarrow \mathcal{X}^s$. As $-K_{\mathcal{X}^{an}}$ is ample on E_1 , $-(\pi' \circ p)^* K_{\mathcal{X}^{an}}$ is nontrivial on the generic fiber of $\hat{E}_1 \rightarrow \text{center}_{\mathcal{X}^s}(E_1)$. This implies $\hat{E}_1 \subset \text{Supp}(E)$ (cf. [KM98, 3.39]). Let the coefficient of \hat{E}_1 in E to be $a > 0$. Then

$$\begin{aligned} \left. \frac{df(\lambda)}{d\lambda} \right|_{\lambda=0} &= (n+1)E \cdot (-p^* K_{\mathcal{X}'})^n \\ &\geq (n+1)a\hat{E}_1 \cdot (-p^* K_{\mathcal{X}'})^n \\ &= (n+1)aE_1 \cdot (-K_{\mathcal{X}^{an}})^n \\ &> 0. \end{aligned}$$

Since $f(\lambda)$ is nondecreasing on $\lambda \in [0, 1]$ and its derivative at 0 is positive, we easily see $f(\lambda) > f(0)$ for any $\lambda \in (0, 1]$. \square

7 Proof of theorems

In this section, we finish proving Theorem 2 by combining the three steps proved in Theorem 3, Theorem 4 and Theorem 5.

Proof of Theorem 2. Let $(\mathcal{X}, \mathcal{L})$ be any polarized generic \mathbb{Q} -Fano family of $(X, -rK_X)$.

Then it follows from Theorem 3 that, after a base change $\phi : C' \rightarrow C$, we get a polarized generic \mathbb{Q} -Fano family $(\mathcal{X}^{lc}, \mathcal{L}^{lc})$ satisfying the properties stated in Theorem 3. After replacing \mathcal{L}^{lc} by its sufficiently large multiples, we can run a sequence of $K_{\mathcal{X}^{lc}}$ -MMP over C with scaling of $\mathcal{H}^{lc} = \mathcal{L}^{lc} - K_{\mathcal{X}^{lc}}$ as in Section 5. We obtain a model $\mathcal{X}^k \rightarrow C'$ with $-K_{\mathcal{X}^k}$ is relatively big and semi-ample. Therefore, it admits an anti-canonical model \mathcal{X}^{an} which satisfies that $(\mathcal{X}^{an}, \mathcal{X}_t^{an})$ is log canonical for every $t \in C'$. Finally, after a base change $C'' \rightarrow C'$, we construct a \mathbb{Q} -Fano family $\mathcal{X}^s \rightarrow C''$. After base change to C'' , all our models after base change are isomorphic over $C'^* \times_C C''$.

In terms of DF intersection numbers,

$$\deg(C'/C) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) \geq \text{DF}(\mathcal{X}^{lc}/C', \mathcal{L}^{lc}) \quad (\text{by Theorem 3}) \quad (11)$$

$$\geq \text{DF}(\mathcal{X}^{an}/C', -rK_{\mathcal{X}^{an}}) \quad (\text{by Theorem 4}) \quad (12)$$

$$\geq \frac{1}{\deg(C''/C')} \text{DF}(\mathcal{X}^s/C'', -rK_{\mathcal{X}^s}) \quad (\text{by Theorem 5}). \quad (13)$$

By Theorem 5, the equality in (13) holds if and only if $\tilde{\mathcal{X}}^{an} = \mathcal{X}^s$. Assume that $t'' \in C''$ is mapped to $t' \in C'$, then $(\mathcal{X}^{an}, \mathcal{X}_{t'}^{an})$ is plt if and only if $(\tilde{\mathcal{X}}^{an}, \tilde{\mathcal{X}}_{t''}^{an})$ (see [KM98, 5.20]) which then implies that \mathcal{X}^{an} is a \mathbb{Q} -Fano family over C' .

By Theorem 4, the equality in (12) holds if and only if $\mathcal{X}^{lc} = \mathcal{X}^{an}$ and $\mathcal{L}^{lc} = \mathcal{L}^{an}$.

Finally by Theorem 3, the equality in (11) holds if and only if $(\mathcal{X}, \mathcal{X}_t)$ is log canonical for any $t \in C$ and $\mathcal{X}^{lc} \cong \mathcal{X} \times_C C'$. As $(\mathcal{X}^{lc}, \mathcal{X}_t^{lc}) \cong (\mathcal{X}^s, \mathcal{X}_t^s)$ is a plt for any $t \in C'$ and $\mathcal{L}^{lc} \sim_{\mathbb{Q}, C'} -rK_{\mathcal{X}^{lc}}$, this implies that $(\mathcal{X}, \mathcal{X}_t)$ is plt for any $t \in C$ (cf. [KM98, 5.20]) and $\mathcal{L} \sim_{\mathbb{Q}, C} -rK_{\mathcal{X}}$, i.e., \mathcal{X} is a \mathbb{Q} -Fano family over C . \square

Proof of Corollary 2. The semistability case follows from Theorem 2 immediately.

Now we assume $(X, -rK_X)$ is K-semistable and all special test configurations $(\mathcal{X}^{st}, \mathcal{L}^{st})$ with $\text{DF}(\mathcal{X}^{st}, \mathcal{L}^{st}) = 0$ satisfy that $\mathcal{X}^{st} \cong X \times \mathbb{A}^1$. Let $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ be an arbitrary normal test configuration whose DF invariant is 0. Applying Theorem 2, we obtain a special test configuration $(\mathcal{X}^{st}, -K_{\mathcal{X}^{st}})$ of $(X, -rK_X)$ and inequalities

$$0 \leq \text{DF}(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}}) \leq m\text{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = 0.$$

Then since the equality holds, by the conclusion of Theorem 2 we know that $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$ is a special test configuration, which implies that $\mathcal{X}^{tc} \cong X \times \mathbb{A}^1$. \square

We finish our paper by the following remark.

Remark 7. *In fact, our general strategy of using the minimal model program to simplify the test configuration and decrease the Donaldson-Futaki invariant works as long as L is proportional to K_X . In the canonically polarized case and the Calabi-Yau case, combining our argument with the separateness of the functors, we can obtain the K-stability in such cases. This was proved algebraically by Odaka in [Oda11]. When X is smooth, this also follows from the existence of Kähler-Einstein metrics (see [Sto09]) after Yau's work [Yau78]). Finally, all the calculation can be almost trivially extended to the log case by the same strategy.*

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